

The initial spectrum of fluctuations in the little bang

Kevin Dusling⁽¹⁾, François Gelis⁽²⁾, Raju Venugopalan⁽³⁾

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1. Physics Department, North Carolina State University,
Raleigh, NC 27695, USA
2. Institut de Physique Théorique (URA 2306 du CNRS), CEA/DSM/Saclay,
91191, Gif-sur-Yvette Cedex, France
3. Physics Department, Bldg. 510A, Brookhaven National Laboratory,
Upton, NY 11973, USA

Abstract

High parton densities in ultra-relativistic nuclear collisions suggest a description of these collisions wherein the high energy nuclear wavefunctions and the initial stages of the nuclear collision are dominated by classical fields. This underlying paradigm can be significantly improved by including quantum fluctuations around the classical background fields. One class of these contributes to the energy evolution of multi-parton correlators in the nuclear wavefunctions. Another dominant class of unstable quantum fluctuations grow rapidly with proper time τ after the collision. These *secular* terms appear at each loop order; the leading contributions can be resummed to all loop orders to obtain expressions for final state observables. The all-order result can be expressed in terms of the spectrum of fluctuations on the initial proper time surface. We compute, in $A^\tau = 0$ gauge, the essential elements in this fluctuation spectrum—the small quantum fluctuation modes in the classical background field. With our derivation in QCD, we have all the ingredients to compute inclusive quantities in heavy ion collisions at early times including i) all-order leading logs in Bjorken $x_{1,2}$ of the two nuclei, ii) all strong multiple scattering contributions, and iii) all-order leading secular terms. In the simpler analogous formalism for a scalar ϕ^4 theory, numerical analysis of the behavior of the energy-momentum tensor is strongly suggestive of early hydrodynamic flow in the system [1]. In QCD, in addition to studying the possible early onset of hydrodynamic behavior, additional important applications of our results include a) the computation of sphaleron transitions off-equilibrium, and b) “jet quenching”, or medium modification of parton spectra, in strong color fields at early times.

1 Introduction

The large flow measured in heavy ion collisions at RHIC [2–5] and more recently at the LHC [6] can be described in hydrodynamic models that have both a nearly *perfect fluid* value of the shear

viscosity to entropy ratio of the quark-gluon matter produced and fairly short thermalization times that usually range between 0.5 and 2 fermis/c [7–9] (depending on the assumptions made about the initial conditions and the implementation of the freeze-out). How isotropization and (subsequently) thermalization is achieved in heavy ion collisions is an outstanding problem which requires that one understand the properties of the relevant degrees of freedom in the nuclear wavefunctions and how these degrees of freedom are released in a collision to produce quark-gluon matter. An *ab initio* approach to the problem can be formulated within the Color Glass Condensate (CGC) effective field theory, which describes the relevant degrees of freedom in the nuclei as dynamical gauge fields coupled to static color sources [10,11]. The computational power of this effective theory is a consequence of the dynamical generation of semi-hard *saturation scale* [12,13] larger than the intrinsic non-perturbative QCD scale ($Q_s^2 \gg \Lambda_{\text{QCD}}^2$), which allows for a weak coupling treatment of the relevant degrees of freedom [14–16] in the high energy nuclear wavefunctions.

There has been significant recent progress in applying the CGC effective field theory to studying the early time behavior of the quark-gluon matter called *Glasma* [17] produced in the initial *little bang* of a high energy heavy ion collision. Inclusive quantities such as the pressure and the energy density in the Glasma can be written as expressions that factorize, to leading logarithmic accuracy in the longitudinal momentum fraction x , the universal properties of the nuclear wavefunctions (measurable for instance in proton-nucleus or electron-nucleus collisions) from the final state evolution of the matter in the collision [18–20]. Key to this approach are the quantum fluctuations around the classical fields. In particular, quantum fluctuations that are invariant under boosts can be shown to factorize into universal density functionals that encode the multi-parton correlations in the nuclear wavefunctions. The evolution of these density functionals with energy is described by the JIMWLK renormalization group equation [21–28].

There are however quantum fluctuations that are not boost invariant. It was observed in [29–31], via numerical solutions (see also [32,33] for a semi-analytic discussion of some instabilities in the solutions Yang-Mills equations) of the classical Yang-Mills equations, that rapidity dependent quantum fluctuations in the expanding Glasma are unstable and grow exponentially as the square root of the proper time τ after the collision. In fact, both the existence and the specific time dependence of these instabilities was anticipated based on studies of the Weibel instabilities in expanding anisotropic Yang-Mills plasmas [34–37]. The unstable quantum fluctuations (initially of order $\mathcal{O}(1)$) become comparable in size to the classical field (of order $\mathcal{O}(g^{-1})$) on a very short time scale $\tau \sim Q_s^{-1}$. Fortunately, one can isolate and resum these rapidly growing *secular* divergences to all orders in perturbation theory. The resulting expressions are free of secular divergences, and can be rephrased as an average over a spectrum of Gaussian fluctuations of the initial data for the classical field encountered at *leading order*. A similar observation was made previously in the context of inflationary cosmology [38,39].

In a previous paper [1], we developed this formalism for a scalar ϕ^4 theory which, like QCD, has a dimensionless coupling in 3+1 dimensions and has unstable modes. We computed the spectrum of fluctuations and showed that the resummed expression for the pressure and energy density can similarly be expressed as an ensemble average over quantum fluctuations. The rapid growth of the unstable fluctuations has drastic consequences. Without resummation, the relation between the energy density and the pressure is not single valued. For the resummed expressions, while the relation between the pressure and energy density is not single valued at early times, it becomes so after a finite time evolution. This development of an *equation of state* therefore allows one to write the conservation equation for the resummed energy momentum tensor $T^{\mu\nu}$ as the closed form set of equations corresponding to the hydrodynamical evolution of a relativistic fluid. This result can be

interpreted as arising from a phase decoherence of the classical field configurations with different initial conditions given by the ensemble of quantum fluctuations. In this theory, the period of the classical trajectories is proportional to the amplitude of the field. Anharmonicity occurs in any non-linear system and we expect the same to hold for QCD. As the different trajectories become phase shifted for different amplitudes there are cancellations resulting in a single valued relation between the pressure and the energy density.

This phenomenon shares several common features with Srednicki's hypothesis of *eigenstate thermalization* and *Berry's conjecture* [40–44]. Berry conjectured in [40] that high lying energy eigenstates of systems whose classical counterpart is chaotic have very complicated wavefunctions that for many purposes behave like random Gaussian¹ functions. A system in such an eigenstate would display features reminiscent of thermal equilibrium, despite being in a pure quantum state [42]. For a system starting initially in a coherent state rather than an energy eigenstate, thermalization would merely amount to losing the initial coherence. Although these ideas were formulated in much simpler systems, they may have some relevance to QCD since here also the underlying classical theory is believed to be chaotic [45,46].

In this paper, we shall focus on computing the initial spectrum of fluctuations in the Glasma formed at early times after a heavy ion collision. The classical background field at $\tau = 0^+$ in the Glasma can be expressed, from the continuity of the Yang-Mills equations across the light-cone [47,48], in terms of classical solutions of the Yang-Mills equations for each of the two nuclei before the collision. For later times, analytical solutions are not known²; however, the Yang-Mills equations have been solved numerically with the initial conditions at $\tau = 0^+$ [51–55]; for a nice review, see [56]. Fortunately, inclusive quantities such as components of the energy-momentum tensor are sensitive only to the initial spectrum of fluctuations about the classical field at $\tau = 0^+$, which can be calculated analytically. Specifically, we will solve the small fluctuations equations of motion in $A^\tau = 0$ gauge, in order to obtain a complete orthonormal basis of these fluctuations. There was a first attempt to compute the small fluctuations 2-point correlator in the Glasma [57] which, as we shall discuss, was incomplete because it did not include fully the structure of the background field.

The paper is organized as follows. In the next section, we will outline the power counting of higher order contributions in the Glasma and emphasize the necessity of resumming secular terms. We isolate the leading contributions and obtain an expression for the resummed leading secular divergences. We show that this expression for inclusive quantities can be rewritten as a path integral over a spectrum of fluctuations times the leading order (classical) expression for the inclusive quantity. The only unknown ingredient in this reformulation are the small fluctuation fields on the initial proper time hypersurface. Additional sub-sections discuss gauge invariance issues and the renormalization of ultraviolet divergences. In section 3, we will show how to compute the small fluctuation fields in the vacuum. We first obtain an inner product for fluctuations on a space-like Cauchy surface that we use to define the orthogonality between a pair of fields. We will further prove that the inner product is independent of the chosen surface. We then show that the small fluctuation fields can be expressed as a linear combination of modes whose coefficients are Gaussian-distributed random complex numbers. (This is equivalent to diagonalizing the small fluctuation correlator on the initial proper time surface.) Because even the computation of the

¹Naturally, the wavefunction of a given eigenstate is not a random function. Berry's conjecture means that for the purpose of computing the expectation value of sufficiently inclusive observables, one can replace the true wavefunction by a random Gaussian function.

²For some interesting recent attempts, see [49,50].

free fluctuations in $A^\tau = 0$ gauge is non-trivial, we shall first solve small fluctuation equations in the vacuum. Then, in the section 4, we shall solve the small fluctuation equations in the background classical field of the Glasma to construct the corresponding physical small fluctuation modes. Section 5 outlines a practical algorithm to compute inclusive quantities (including all leading logarithms in x , and all leading secular contributions) as a function of proper time. As we shall demonstrate, the complexity of this space-time evolution is manageable, and amounts to diagonalizing certain matrices on the initial proper time surface. In the final section, we re-state our key results and discuss some important applications. These include a) a systematic study of possible thermalization of the quantum system whose evolution with proper time we plan to simulate numerically just as for the scalar case studied previously. b) The nature and role of sphaleron transitions in the early time dynamics of the system. c) The medium modification of hard probes to study 'jet quenching' at early times. An open question for future work is to explore until what times these analysis are valid and how one can incorporate sub-leading contributions that become increasingly important at late times. There are two appendices. The first discusses technical aspects of the computation of small fluctuation fields in the vacuum. Expressions for the Wightman functions for free fields in $A^\tau = 0$ gauge are discussed in the second appendix, where some connections to previous work on these is also discussed [57,58].

2 Resummation of leading instabilities in the Glasma

We will begin by first outlining how the power counting for computing inclusive quantities in field theories with strong time dependent sources is modified due to the presence of secular divergences. Following this power counting, we derive an explicit expression for the energy-momentum tensor in heavy ion collisions that resums the leading instabilities to all loop orders in perturbation theory. We will show that the resummed expression for the energy momentum tensor can be expressed as a path integral over the product of two terms. The first is a weight functional that samples the spectrum of quantum fluctuations on the initial proper time hypersurface, while the second is the leading order expression for the energy-momentum tensor. In the latter, the classical field is shifted by the sampled quantum fluctuations. Computing the initial spectrum of fluctuations is our primary goal in this paper, the derivation of which will be discussed at length in sections 3 and 4. Before we go there, two further sub-sections will discuss the constraints imposed by gauge invariance on the spectrum of fluctuations and the nature of ultraviolet divergences respectively.

2.1 Power counting of unstable modes in the Glasma

In previous works [59,60], it was shown that the problem of computing leading order (LO) and next-to-leading order (NLO) contributions to *inclusive* quantities –such as components of the energy momentum tensor– in field theories with strong time dependent sources can be formulated as an initial value problem where a classical field determined on an initial Cauchy surface is evolved up to the time at which the (local) observable is computed. Because one anticipates that a semi-hard scale $Q_s^2 \gg \Lambda_{\text{QCD}}^2$ is generated by the non-linear QCD dynamics at high energy [12,13], a systematic weak coupling expansion of these inclusive quantities is feasible. One can formally arrange the perturbative expansion of an observable such as the energy momentum tensor as

$$\mathcal{O}[\rho_1, \rho_2] = \frac{1}{g^2} \left[c_0 + c_1 g^2 + c_2 g^4 + \dots \right], \quad (1)$$

where each term corresponds to a different loop order. Each of the coefficients c_n is itself an infinite series of terms involving arbitrary orders in $(g\rho_{1,2})^p$. These terms are all of order unity because the color charge densities are of order $\rho_{1,2} \sim \mathcal{O}(g^{-1})$ in a large nucleus at high energy. The color charge densities correspond to the large x color sources in either nucleus 1 or nucleus 2 respectively in a heavy ion collision. Their evolution with the separation scale between sources and fields is described by the JIMWLK equation, which will be stated shortly. The LO contribution comes from the first coefficient c_0 ,

$$\mathcal{O}_{\text{LO}}[\rho_1, \rho_2] \equiv \frac{c_0}{g^2} . \quad (2)$$

This leading term c_0/g^2 has been studied extensively for the single inclusive gluon distribution in A+A collisions [51–54] and recently for the double inclusive distribution as well [61].

Following this terminology, we denote

$$\mathcal{O}_{\text{NLO}}[\rho_1, \rho_2] \equiv c_1 , \quad \mathcal{O}_{\text{NNLO}}[\rho_1, \rho_2] \equiv c_2 g^2 , \quad \dots \quad (3)$$

At each order in the loop expansion, there can arise contributions from the loop integrals which are of the same magnitude as lower orders. One set of such contributions are the increasingly large logarithms of the momentum fractions $x_{1,2}$ of partons in the nuclear wavefunctions as higher energies, or equivalently smaller values of $x_{1,2}$, are achieved in nuclear collisions. The term c_n can have up to n powers of such logarithms, with *leading logarithmic* terms identified as terms that have as many logarithms as their order in the loop expansion,

$$\mathcal{O}_{\text{LLog}}[\rho_1, \rho_2] \equiv \frac{1}{g^2} \sum_{n=0}^{\infty} d_n \left[g^2 \ln \left(\frac{1}{x_{1,2}} \right) \right]^n , \quad (4)$$

where d_n is the coefficient of n -th term in the leading log expansion. We were able to show [18–20] that the leading logarithmic contributions in $x_{1,2}$, after averaging over the sources $\rho_{1,2}$ factorize into the expression

$$\langle \mathcal{O} \rangle_{\text{LLog}} = \int [D\rho_1 D\rho_2] W_{x_1}[\rho_1] W_{x_2}[\rho_2] \mathcal{O}_{\text{LO}}[\rho_1, \rho_2] , \quad (5)$$

where $W_{x_{1,2}}[\rho_{1,2}]$ are the density functionals we alluded to previously. These obey the JIMWLK equation [21–28]

$$\frac{\partial W_{x_{1,2}}[\rho_{1,2}]}{\partial \ln(1/x_{1,2})} = \mathcal{H}_{1,2} W_{x_{1,2}}[\rho_{1,2}] . \quad (6)$$

Here $\mathcal{H}_{1,2}$ are the JIMWLK Hamiltonians of the two nuclei; since their explicit form is not essential to the discussion here, we will refer the interested reader to ref. [18] for explicit expressions in our notation. Given an initial condition at some initial x_0 value, the JIMWLK equation describes the evolution in the nuclear wavefunctions of the multi-parton correlators that contribute to inclusive observables measured in the final state.

The resummation of quantum corrections arising from logarithms in $x_{1,2}$, as sketched here, takes into account contributions that are essential in describing the energy evolution of inclusive observables in heavy ion collisions. These contributions are zero modes in ν , the Fourier conjugate of the space-time rapidity η , and are localized in rapidity around the wave functions of the incoming nuclei. There are also quantum fluctuations that are non-zero modes of ν . Such contributions, that

do not bring leading logs of $1/x_{1,2}$, cannot be factorized into the evolution of the density functionals $W_{x_{1,2}}$ in eq. (5). As shown in [29–31,62], these terms can be unstable and grow exponentially with the square root of the proper time (equal to $\tau \equiv \sqrt{2x^+x^-}$ in light-cone co-ordinates) for a system undergoing one dimensional longitudinal expansion. Based on these considerations, the expansion we sketched in eqs. (1) and (4) needs to be modified to keep track also of quantum fluctuations of amplitude $g \exp(\sqrt{\mu\tau})$ (where μ is a growth rate of order Q_s) relative to the leading term. This is necessary because the rapid growth of these unstable modes leads to a break down of the perturbative expansion when

$$\tau \sim \tau_{\max} \equiv \mu^{-1} \ln^2 \left(\frac{1}{g} \right) \quad (7)$$

is reached, the proper time at which 1-loop corrections become as large as the leading order term. The breakdown of the expansion can be avoided if one resums these divergent contributions, leading to a resummed result that is well behaved for $\tau \rightarrow +\infty$. Taking into account both the leading logs in $1/x_{1,2}$ and the leading unstable contributions, the new expansion reads

$$\mathcal{O}_{\text{LLog}}^{\text{LLog}} [\rho_1, \rho_2] \equiv \frac{1}{g^2} \sum_{n=0}^{\infty} g^{2n} \sum_{p+q=n} \tilde{d}_{p,q} \ln^p \left(\frac{1}{x_{1,2}} \right) e^{2q\sqrt{\mu\tau}}. \quad (8)$$

Thus far, we have only resummed the $q = 0$ sector of this formula, where the result of the resummation is expressed by the factorized formula (5). The two sources of leading quantum fluctuations at this accuracy can be resummed independently because a given quantum fluctuation mode cannot be at the same time a zero mode (that generates logarithms in $x_{1,2}$) and a non-zero mode (that generates a secular divergence in proper time τ). Naturally, in higher loop corrections, one loop can bring a log of $1/x_{1,2}$ while another loop brings a secular divergence. This is why eq. (8) has terms with both p and q non-zero simultaneously. But the independence of the two types of divergences, based ultimately on considerations of causality, leads us to expect that the double series of eq. (8) can be factorized into a series in p times a series in q .

2.2 All orders resummation of the leading secular terms

We shall now discuss how resumming the leading secular terms modifies the expression of eq. (5). Albeit our considerations apply to any inclusive quantity, for specificity, we shall consider here the energy-momentum tensor.

2.2.1 Reminder of LO and NLO results

Let us recall first that at leading order in g^2 , the energy-momentum tensor $T_{\text{LO}}^{\mu\nu}$ is given by

$$T_{\text{LO}}^{\mu\nu}(x) = \frac{1}{4} g^{\mu\nu} \mathcal{F}_a^{\alpha\beta} \mathcal{F}_{a,\alpha\beta} - \mathcal{F}_a^{\mu\alpha} \mathcal{F}_{a\alpha}^{\nu}, \quad (9)$$

with the field strength tensor defined as

$$\mathcal{F}_a^{\mu\nu} = \partial^\mu \mathcal{A}_a^\nu - \partial^\nu \mathcal{A}_a^\mu + g f^{abc} \mathcal{A}_b^\mu \mathcal{A}_c^\nu, \quad (10)$$

where \mathcal{A}_a^μ is the solution of the classical Yang-Mills³ equations with sources $\rho_{1,2}$ that vanishes at $x^0 \rightarrow -\infty$,

$$\mathcal{D}_\mu^{ab} \mathcal{F}_b^{\mu\nu} = \delta^{\nu+} \rho_1^a + \delta^{\nu-} \rho_2^a \quad , \quad \lim_{x^0 \rightarrow -\infty} \mathcal{A}_a^\mu(x) = 0 \quad . \quad (11)$$

One can then express the NLO contribution to the energy-momentum tensor, for a given distribution of color sources⁴ and at an arbitrary space-time point, as the action of a functional operator acting on the LO result [18,59],

$$T_{\text{NLO}}^{\mu\nu}(x) = \left[\int_\Sigma d^3\mathbf{u} \, \beta \cdot \mathbb{T}_\mathbf{u} + \frac{1}{2} \int_\Sigma d^3\mathbf{u} \, d^3\mathbf{v} \sum_{\lambda,a} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k} [a_{+\mathbf{k}\lambda a} \cdot \mathbb{T}_\mathbf{u}] [a_{-\mathbf{k}\lambda a} \cdot \mathbb{T}_\mathbf{v}] \right] T_{\text{LO}}^{\mu\nu}(x) \quad , \quad (12)$$

where Σ is a Cauchy surface where the initial values of the classical color field and its derivatives are specified. In this formula, β and $a_{\pm\mathbf{k}\lambda a}$ are small corrections to the gauge field \mathcal{A}^μ . More specifically, $\beta(\mathbf{u})$ is the one loop correction to the classical field on the surface Σ (see [18] for more details). In applications to heavy ion collisions, a natural choice of Σ is the surface at proper time $\tau = 0^+$, which corresponds physically to times just after the two nuclei have collided. Though $\tau = 0^+$ is the initial surface of choice, we will at the outset consider a generic space-time hypersurface. The only constraint on Σ is that, for the forthcoming resummation to be effective, it must be located at times before the unstable modes have become too large⁵. In the right hand side of eq. (12), $\mathbb{T}_\mathbf{u}$ is the generator of shifts of the initial data for the classical field on Σ . Generically, it reads⁶

$$a \cdot \mathbb{T}_\mathbf{u} = a^\mu(\mathbf{u}) \frac{\delta}{\delta \mathcal{A}^\mu(\mathbf{u})} + (\partial^\nu a^\mu(\mathbf{u})) \frac{\delta}{\delta (\partial^\nu \mathcal{A}^\mu(\mathbf{u}))} \quad , \quad (13)$$

where $\mathcal{A}^\mu(\mathbf{u})$ (in curly font and without a time argument) is the value of the classical field on the initial time surface. Note that in general, specific gauge conditions and specific choices of the surface Σ reduce the number of terms this operator contains. In its minimal form, it contains one term for each independent field component or field derivative component that one must specify in the initial value problem on Σ .

Let us now explicit a bit more the fields $a_{\pm\mathbf{k}\lambda a}^\mu$ that appear in eq. (12). They are small fluctuation fields about the classical field \mathcal{A}^μ , that obey the equation of motion⁷

$$\mathcal{D}_\mu (\mathcal{D}^\mu a^\nu - \mathcal{D}^\nu a^\mu) - ig \mathcal{F}^\nu{}_\mu a^\mu = 0 \quad , \quad (14)$$

³We have written the Yang-Mills equations in a form that involves the adjoint representation of the covariant derivative, $\mathcal{D}_\mu^{ab} = \partial_\mu \delta^{ab} - ig \mathcal{A}_\mu^{ab}$, where \mathcal{A}_μ^{ab} is the classical gauge potential in the adjoint representation. It is important to distinguish the \mathcal{A}_μ^{ab} 's from the \mathcal{A}_μ^a 's that are the components of the SU(3) element \mathcal{A}_μ in its decomposition over the generators of the algebra, $\mathcal{A}_\mu \equiv \mathcal{A}_\mu^a t^a$. The two sets of coefficients are related by $\mathcal{A}_\mu^{bc} = -if^{abc} \mathcal{A}_\mu^a$, since the components of the generators in the adjoint representation are $(t_{\text{adj}}^a)_{bc} = -if^{abc}$.

⁴Unless specified otherwise, the dependence on ρ_1, ρ_2 , the color charge densities in each of the nuclei, will be implicit in our discussion.

⁵The NLO expression in eq. (12) does not depend on the choice of Σ . However, our resummed result will depend on this choice since it includes only a subset of the higher loop corrections. Provided the surface Σ is located in a region where the unstable fluctuations are still small, the difference between various choices of Σ is a small correction. We will discuss this point further later in the paper.

⁶It has dimension of (mass)² because $\dim[\frac{\delta}{\delta \mathcal{A}}] = (\text{mass})^2$ and $\dim[a_{\pm\mathbf{k}\lambda a}] = (\text{mass})^0$.

⁷To avoid cumbersome notations, we have not written explicitly the color indices of the various objects. Here, and henceforth, the \mathcal{D} 's and \mathcal{F} should be understood as objects in the adjoint representation. For instance $\mathcal{F}^\nu{}_\mu a^\mu$ means $\mathcal{F}_{ab\mu}^\nu a_b^\mu$. Likewise, $\mathcal{D}_\mu \mathcal{D}^\mu a^\nu$ is $\mathcal{D}_{ab,\mu} \mathcal{D}_{bc}^\mu a_c^\nu$.

and that are specified in the remote past by the boundary condition

$$\lim_{x^0 \rightarrow -\infty} a_{\pm \mathbf{k} \lambda a}^\mu(x) = \varepsilon_{\mathbf{k} \lambda}^\mu T^a e^{\pm i \mathbf{k} \cdot \mathbf{x}} . \quad (15)$$

The T^a 's are the SU(3) generators and $\varepsilon_{\mathbf{k} \lambda}^\mu$ is the polarization vector. Thus the labels \mathbf{k}, λ, a are respectively the initial momentum, initial polarization and initial color of the gauge fluctuation represented by $a_{\pm \mathbf{k} \lambda a}$, and the sign \pm specifies whether it is a positive or negative energy wave in the remote past.

2.2.2 Power counting rules

At leading order (tree level), the energy momentum tensor is of order Q_s^4/g^2 . In the absence of secular divergences, from the power counting described previously, the NLO corrections should be of order Q_s^4 . This power counting could be obtained in eq. (12) by noting that

$$a_{\pm \mathbf{k} \lambda a} \sim \mathcal{O}(1) , \quad (16)$$

$$\beta \sim \mathcal{O}(g) , \quad (17)$$

$$\mathbb{T}_{\mathbf{u}} \sim \frac{\delta}{\delta \mathcal{A}} \sim \mathcal{O}(g) , \quad (18)$$

since $\mathcal{A} \sim \mathcal{O}(g^{-1})$. The existence of instabilities implies that we must alter our estimate of the order of magnitude of the operators $\mathbb{T}_{\mathbf{u}}$. Indeed, since $\mathbb{T}_{\mathbf{u}} \mathcal{A}(\tau, \mathbf{x})$ is the propagator of a small fluctuation over the background field between a point on the initial proper time surface and the point (τ, \mathbf{x}) , it grows at the same pace as the unstable fluctuations. Thus the counting rule for $\mathbb{T}_{\mathbf{u}}$ should be modified to read

$$\mathbb{T}_{\mathbf{u}} \sim \mathcal{O}(g e^{\sqrt{\mu \tau}}) . \quad (19)$$

The combination $\mathbb{T}_{\mathbf{u}} \mathbb{T}_{\mathbf{v}}$ in eq. (12) then grows as $g^2 \exp(2\sqrt{\mu \tau})$ which leads to a break down of the power counting at the proper time τ_{\max} defined in eq. (7). At τ_{\max} , the 1-loop correction becomes as large as the leading order contribution, and one may anticipate that an infinite series of higher loop corrections also become equally important at this time.

2.2.3 Selection of the leading terms

Our goal is now to collect from higher orders all the terms that are leading at the time τ_{\max} . This comprises all the terms where the extra powers of g^2 are compensated by an equal number of factors of $e^{2\sqrt{\mu \tau}}$. We presume that a typical higher order correction to the energy momentum tensor can still be written in the form of eq. (12), but with a more general operator acting on $T_{\text{LO}}^{\mu\nu}(x)$ of the form

$$\int_{\Sigma} d^3 \mathbf{u}_1 \cdots d^3 \mathbf{u}_n \Gamma_n(\mathbf{u}_1, \cdots, \mathbf{u}_n) \cdot \mathbb{T}_{\mathbf{u}_1} \cdots \mathbb{T}_{\mathbf{u}_n} . \quad (20)$$

Here Γ_n is an n -point function, which may or may not be simply connected. This expression has not been proven in general but results from a conjecture that inclusive quantities at all loop orders can be expressed purely in terms of retarded propagators, thereby generalizing known results at

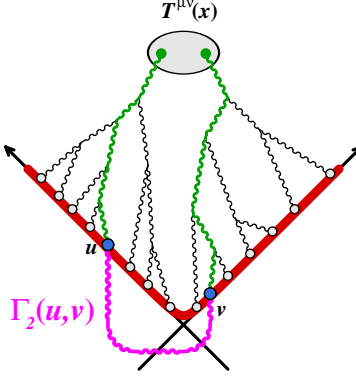


Figure 1: Representation of the 1-loop contribution involving the function $\Gamma_2(\mathbf{u}, \mathbf{v})$. The thick red line is the $\tau = 0^+$ surface on which the initial value problem is set up. The open circles represent the initial data. The filled blue circles represent the two operators $\mathbb{T}_{\mathbf{u}, \mathbf{v}}$, and the U-shaped wavy line represented below the light-cone is the function $\Gamma_2(\mathbf{u}, \mathbf{v})$.

LO and NLO. While there are specific examples of loop contributions that have been checked to satisfy this conjecture, there are in particular nested loops contributions for which the conjecture is difficult to confirm. In the figures 1 and 2, we illustrate this formula by some examples of 1-loop and 2-loop contributions.

With the stated assumption implicit in eq. (20), the following power counting can be established. If eq. (20) is a piece of a L -loop correction to the energy-momentum tensor, the order g^p and the number n of points of the function Γ_n are related by

$$n + p = 2L. \quad (21)$$

This formula can be checked for the examples of graphs given in the figures 1 and 2. Note that $p = 0$ is the smallest possible value for p . Taking into account the effect of the instabilities (i.e. one power of $\exp(\sqrt{\mu\tau})$ for each of the n operators $\mathbb{T}_{\mathbf{u}_i}$), the order of magnitude of a contribution obtained from eq. (20) is

$$g^p \left[g e^{\sqrt{\mu\tau}} \right]^n, \quad (22)$$

relative to the leading order contribution. If we just count naively the powers of g , the power counting would indicate that this contribution gives a correction of order g^{2L} , a decrease by a factor g^2 for each extra loop. However, because of the instability, by the time τ_{\max} given in eq. (7), the contribution is instead of order g^p and does not depend anymore on the number n of \mathbb{T} operators. At the time τ_{\max} all the terms with $p = 0$, regardless of the number of loops, are of the same order while all the remaining terms for which $p > 0$ are suppressed by additional powers of g . It is therefore natural to resum all the $p = 0$ terms, and to neglect all those with $p > 0$ as giving sub-leading contributions. This implies that the numbers n of \mathbb{T} operators must be even and equal to $2L$. Moreover, since we keep only terms of order $p = 0$ in Γ_{2L} , the only possibility that remains⁸ is to construct Γ_{2L} as a product of L factors Γ_2 (an example of which is the left

⁸Note that tadpole contributions such as the term $\beta \cdot \mathbb{T}_{\mathbf{u}}$ in eq. (12) are also excluded since $\beta \sim \mathcal{O}(g)$.

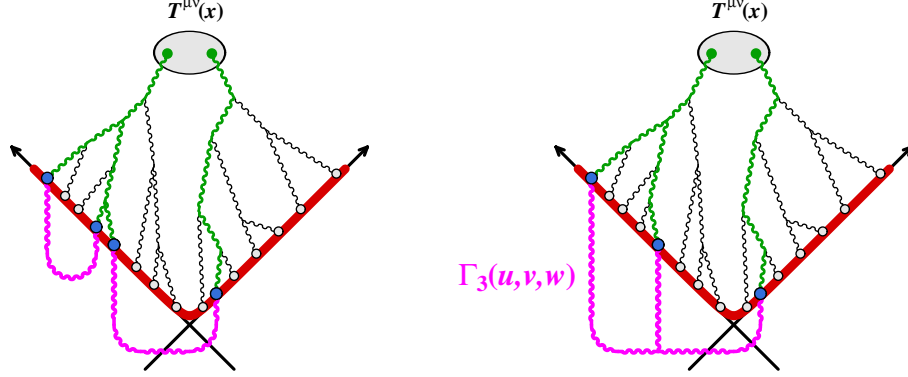


Figure 2: Representation of two examples of 2-loop contributions. The thick red line is the $\tau = 0^+$ surface on which the initial value problem is set up. The open circles represent the initial data. The filled blue circles represent operators $\mathbb{T}_{\mathbf{u}, \mathbf{v}}$. Left: contribution with a Γ_4 that factorizes into two Γ_2 's. Right: contribution with a Γ_3 .

diagram of figure 2) because any non-factorized contribution to Γ_{2L} requires more powers of g .

2.2.4 Resummation: formal expression

Therefore, the leading operator at L loops in eq. (20) is the L -th power of the 2-point operator that appears at 1-loop,

$$\frac{1}{L!} \left[\frac{1}{2} \int_{\Sigma} d^3 \mathbf{u} d^3 \mathbf{v} \Gamma_2(\mathbf{u}, \mathbf{v}) \cdot \mathbb{T}_{\mathbf{u}} \mathbb{T}_{\mathbf{v}} \right]^L, \quad (23)$$

where

$$\Gamma_2(\mathbf{u}, \mathbf{v}) \cdot \mathbb{T}_{\mathbf{u}} \mathbb{T}_{\mathbf{v}} = \sum_{\lambda, a} \int \frac{d^3 \mathbf{k}}{(2\pi)^3 2k} [a_{+\mathbf{k}\lambda a} \cdot \mathbb{T}_{\mathbf{u}}] [a_{-\mathbf{k}\lambda a} \cdot \mathbb{T}_{\mathbf{v}}]. \quad (24)$$

The inverse factorial prefactor is a symmetry factor that prevents multiple counting when the various factors Γ_2 are permuted. Summing all the contributions from $L = 0$ (leading order) to $L = +\infty$, we obtain⁹

$$T_{\text{resummed}}^{\mu\nu}(x) = \exp \left[\frac{1}{2} \int_{\Sigma} d^3 \mathbf{u} d^3 \mathbf{v} \Gamma_2(\mathbf{u}, \mathbf{v}) \cdot \mathbb{T}_{\mathbf{u}} \mathbb{T}_{\mathbf{v}} \right] T_{\text{LO}}^{\mu\nu}(x). \quad (25)$$

⁹Until the conjecture in eq. (20) is proved, one may choose to interpret eq. (25) as a well motivated ansatz resulting from the exponentiation of the NLO result.

Eq. (25) provides an expression that resums all the leading contributions¹⁰ at the time $\tau = \tau_{\max}$. However, this expression is very formal as it is expressed in terms of functional derivatives with respect to the initial conditions for the classical color fields and their time derivatives on the initial proper time surface Σ . Fortunately, as we shall see, we can rewrite this result in a form that is much more transparent both conceptually and for computational purposes.

2.2.5 Resummation: path integral representation

We first recall that the operator $\mathbb{T}_{\mathbf{u}}$ defined in eq. (13) is the generator of shifts of the initial conditions at all points \mathbf{u} on the initial proper time surface $\tau = 0^+$ for the classical fields \mathcal{A}^μ and their time derivatives $\partial_\tau \mathcal{A}^\mu$, the latter either being equal to or simply proportional to the corresponding electric fields, their canonical conjugate momenta. We can therefore write

$$\exp \left[\int_{\Sigma} d^3 \mathbf{u} [\alpha \cdot \mathbb{T}_{\mathbf{u}}] \right] \mathcal{F}[\mathbb{A}] = \mathcal{F}[\mathbb{A} + \alpha] , \quad (26)$$

where $\mathbb{A} \equiv (\mathcal{A}, \mathcal{E})$ denotes collectively all the components of the initial classical field and their canonically conjugate momenta on the initial time surface. One should similarly understand α to denote small perturbations of both the initial classical field and their canonically conjugate electric fields. We next obtain¹¹

$$\exp \left[\frac{1}{2} \int_{\Sigma} d^3 \mathbf{u} d^3 \mathbf{v} \Gamma_2(\mathbf{u}, \mathbf{v}) \cdot \mathbb{T}_{\mathbf{u}} \mathbb{T}_{\mathbf{v}} \right] = \int [\mathcal{D}\alpha] F_0[\alpha] \exp \left[\int_{\Sigma} d^3 \mathbf{u} [\alpha \cdot \mathbb{T}_{\mathbf{u}}] \right] , \quad (27)$$

with¹²

$$F_0[\alpha] \propto \exp \left[- \frac{1}{2} \int_{\Sigma} d^3 \mathbf{u} d^3 \mathbf{v} \alpha(\mathbf{u}) \Gamma_2^{-1}(\mathbf{u}, \mathbf{v}) \alpha(\mathbf{v}) \right] . \quad (28)$$

In eq. (27), the functional integration $[D\alpha]$ is also a shorthand for integrations over all the components of the perturbation and of its time derivative on the initial surface.

¹⁰Subleading contributions such as the $p = 1$ contribution $g \cdot g \exp(\sqrt{\mu\tau})$ are no longer sub-leading by times $\tau \sim \mu^{-1} \ln^2(g^{-2})$. This time is only slightly larger than the time τ_{\max} at which the $p = 0$ terms become important. Therefore, including one by one the $p = 1$ terms on top of an expression that resums the $p = 0$ terms is bound to fail – these contributions must be included all at once via a resummation. Having this in mind, a more important question is: do these contributions ever become important after they have been appropriately resummed? A resummation of the $p = 1$ secular terms is outside the scope of this work, but it is plausible that they can be included in our framework by a modification of the distribution $F_0[\alpha]$ of the fluctuations at the initial time. If this is the case, then the $p > 1$ terms would simply lead to small non-gaussianities in the spectrum of fluctuations. However, the is at present nothing more than a conjecture, and the results of this paper have to be interpreted with this in mind.

¹¹An elementary form of this identity,

$$e^{\frac{\alpha}{2} \partial_x^2} f(x) = \int_{-\infty}^{+\infty} dz \frac{e^{-z^2/2\alpha}}{\sqrt{2\pi\alpha}} f(x+z) ,$$

can be proved by performing a Taylor expansion of the exponential on the left hand side and of $f(x+z)$ on the right hand side of this expression. From this simple example, one sees that a Gaussian operator in derivatives is a *smearing operator* that convolutes the target function with a Gaussian distribution.

¹²The unwritten constant prefactor, proportional to $[\det(\Gamma_2)]^{-1/2}$, is such that the distribution $F_0[\alpha]$ has an integral over α normalized to unity.

With the identities (26) and (27) in hand, we can rewrite our formal result in eq. (25) as

$$T_{\text{resummed}}^{\mu\nu}(x) = \int [\mathcal{D}\alpha] F_0[\alpha] T_{\text{LO}}^{\mu\nu}[\mathbb{A} + \alpha](x) , \quad (29)$$

where the weight functional F_0 , corresponding to the initial spectrum of fluctuations, is defined in eq. (28). This result is a central expression of our paper and was sketched previously in [18,63]. It was also obtained in previous works for a scalar field theory [39] and a gauge field theory [57] respectively using different methods. The expression is quite remarkable because it demonstrates that the resummation of loop (quantum) corrections that correspond to the most unstable configurations in a single heavy ion collision event can be expressed as an average over a Gaussian distributed ensemble of classical configurations in the Glasma. Note also that, although this formula was derived here for the energy-momentum tensor, the power counting that led us to the exponentiation of the 1-loop result did not depend on the choice of a specific observable. Thus we expect that the same resummation would also be applicable to other inclusive quantities; the spectrum of fluctuations superposed on the Glasma fields is universal.

The essential ingredient in eq. (29) is the small fluctuations correlator $\Gamma_2(\mathbf{u}, \mathbf{v})$, defined in eq. (24), which should be computed with the two endpoints on the initial time surface Σ , with the Glasma field in the background. A first attempt to compute this object is given in [57]. However, the expression obtained there was incomplete because it corresponds to an approximate expression for the free propagator in $A^\tau = 0$ gauge, with the only dependence on the background field coming from Gauss's law. We will compute here (in $A^\tau = 0$ gauge) the spectrum of fluctuations both in the free case and in the presence of a classical background field. We will show later that the latter has a non-trivial dependence on the classical fields in the Glasma at Σ .

2.3 Gauge invariance of the spectrum of fluctuations

The left hand side of the expression in eq. (29) should be gauge invariant because the energy-momentum tensor is a physical quantity. It should therefore be invariant under a gauge transformation of the classical Glasma field,

$$\mathcal{A} \longrightarrow \Omega^\dagger \mathcal{A}^\mu \Omega + \frac{i}{g} \Omega^\dagger \partial^\mu \Omega . \quad (30)$$

This invariance is also true for its leading order counterpart on the right hand side of the expression, when we apply the same gauge transformation to the total field $\mathcal{A} + \alpha$,

$$\mathcal{A} + \alpha \longrightarrow \Omega^\dagger (\mathcal{A}^\mu + \alpha^\mu) \Omega + \frac{i}{g} \Omega^\dagger \partial^\mu \Omega . \quad (31)$$

Thus for the expression in eq. (29) to be manifestly gauge invariant, it is sufficient if the initial spectrum of fluctuations F_0 is invariant under the transformation

$$\alpha \longrightarrow \Omega^\dagger \alpha \Omega . \quad (32)$$

If we decompose α on the basis of the generators of $SU(3)$, $\alpha \equiv \alpha_a t^a$, the identity

$$\bar{\Omega}_{ab} t^b = \Omega t^a \Omega^\dagger , \quad (33)$$

(where $\bar{\Omega}_{ab}$ here is an adjoint SU(3) matrix) gives the equivalent transformation for the components α_a to be

$$\alpha_a \rightarrow \bar{\Omega}_{ab} a^b \equiv \tilde{\alpha}_a . \quad (34)$$

For the argument of the exponential in F_0 to be invariant under this transformation,

$$\alpha_a(\mathbf{u}) \Gamma_{2,ab}^{-1}(\mathbf{u}, \mathbf{v}) \alpha_b(\mathbf{v}) \longrightarrow \tilde{\alpha}_a(\mathbf{u}) \tilde{\Gamma}_{2,ab}^{-1}(\mathbf{u}, \mathbf{v}) \tilde{\alpha}_b(\mathbf{v}) , \quad (35)$$

the inverse small fluctuations correlator in the Glasma (which is an 8×8 adjoint matrix in SU(3)) must satisfy

$$\tilde{\Gamma}_2^{-1}(\mathbf{u}, \mathbf{v}) = \bar{\Omega}(\mathbf{u}) \Gamma_2^{-1}(\mathbf{u}, \mathbf{v}) \bar{\Omega}^\dagger(\mathbf{v}) . \quad (36)$$

It is clear from the structure of the expression in eq. (24) that this property will be satisfied.

2.4 Renormalization of ultraviolet divergences in the Glasma

It is also important to address the potential ultraviolet divergences in eq. (29). The leading order energy-momentum tensor in the Glasma is ultraviolet finite. However, because one is resumming quantum fluctuations in eq. (29), the energy-momentum tensor should receive a contribution from the (infinite) zero point energy. One can regularize ultraviolet divergences by introducing a cutoff Λ corresponding to the largest momentum mode¹³ of the fluctuation α . Since the energy-momentum tensor has canonical dimension four, we expect that its dependence on this cutoff can be organized as

$$T_{\text{resummed}}^{\mu\nu}(x) = c_1 \Lambda^4 + c_2 \Lambda^2 + c_3 , \quad (37)$$

where $c_{1,2,3}$ are finite quantities. It is easy to renormalize the energy-momentum tensor by subtraction if one can prove that the divergences are truly a property of the vacuum and do not depend on the background classical field \mathcal{A} in the Glasma. The coefficient c_1 is dimensionless – it is therefore a pure number, that cannot depend on the background field. The case of c_2 is trickier. Indeed, its canonical dimension 2 allows a priori a dependence on the background field. However, we know that the left hand side in eq. (37) is invariant under gauge transformations of the background field; we therefore must conclude that the coefficient c_2 must be a gauge invariant, local (because the left hand side is a local quantity), dimension 2 quantity. There is no such quantity in Yang-Mills theory, which suggests that $c_2 = 0$. Thus, on the basis of gauge symmetry and locality, we expect that the only ultraviolet divergence in our expression for the resummed energy-momentum tensor is a quartic divergence, with a coefficient that does not depend on the background field. It can be computed in principle once and for all in the absence of the background field and subtracted from $T_{\text{resummed}}^{\mu\nu}$ to give an ultraviolet finite result for this quantity.

3 Orthonormal basis of small fluctuations

3.1 Introduction

We noted in eq. (29) of the previous section that the small fluctuations 2-point correlator $\Gamma_2(\mathbf{u}, \mathbf{v})$ is the key ingredient in resumming the contributions of leading instabilities at all loop orders to the energy-momentum tensor.

¹³In practical implementations of this resummation, space is discretized on a lattice, and thus the UV cutoff is the inverse of the lattice spacing.

In this section and the following one, we will work in the *Fock-Schwinger* gauge $A^\tau = 0$. Albeit at first sight a natural gauge for describing hadron-hadron collisions (because it is an interpolation between two light-cone gauges in the forward light-cone), the *Fock-Schwinger* gauge is not frequently used in the literature. This is because even expressions for the free correlator are complicated in this gauge. Our motivation here is specific to the nature of the CGC description of heavy ion collisions. The initial conditions for the evolution of classical gauge fields in the forward light cone, in this gauge, are simply expressed [47] in terms of the classical fields in the nuclei before the collision. This is an important criterion because our resummed result in eq. (29) for the energy-momentum tensor is expressed in terms of solutions of classical Yang-Mills equations with Gaussian distributed initial conditions. Further, numerical computations are unavoidable because one is in a strong field regime where perturbative computations are invalid; thus analytically cumbersome expressions are not a deterrent if efficient numerical algorithms are feasible.

Turning to the computation of the small fluctuations correlator,

$$\Gamma_2(\mathbf{u}, \mathbf{v}) = \sum_{\lambda, a} \int \frac{d^3 \mathbf{k}}{(2\pi)^3 2k} a_{+\mathbf{k}\lambda a}(\mathbf{u}) a_{-\mathbf{k}\lambda a}(\mathbf{v}) , \quad (38)$$

as noted previously in the discussion after eq. (13), the fluctuation fields $a_{\pm\mathbf{k}\lambda a}^\mu$ are plane wave fields at $x^0 = -\infty$ that have been evolved in time by interacting with the classical background field \mathcal{A} . Before going further, let us state two properties of this correlator that are true when the two points \mathbf{u} and \mathbf{v} lie on the same Cauchy surface¹⁴,

- i. Γ_2 is symmetric: $\Gamma_2(\mathbf{u}, \mathbf{v}) = \Gamma_2(\mathbf{v}, \mathbf{u})$,
- ii. Γ_2 is real valued.

Note that for the correlator as defined, the two properties are equivalent since $a_{-\mathbf{k}\lambda a} = (a_{+\mathbf{k}\lambda a})^*$. These properties are crucial since Γ_2 is the variance of the fluctuations in our resummation formula, eq. (29).

From the definition given in eq. (38), one might presume that the calculation of Γ_2 requires one to follow the entire evolution from $x^0 = -\infty$ to $\tau = 0^+$. We will demonstrate in section 3.3 that this is not the case; to compute Γ_2 , it is sufficient to construct an orthonormal basis of fluctuation fields about classical fields at small proper times $\tau = 0^+$.

3.2 Curvilinear coordinates

The fluctuations $a_{\pm\mathbf{k}\lambda a}^\mu$ start as plane waves in the remote past and evolve about the Glasma classical field \mathcal{A} according to the equation of motion (14). This expression of the equation of motion, as well as the Yang-Mills equation (11), implicitly assume a Cartesian system of coordinates. However, the most natural co-ordinate system in the treatment of the post-collision evolution is the $(\tau, \eta, \mathbf{x}_\perp)$ system, where

$$\tau \equiv \sqrt{t^2 - z^2} \quad , \quad \eta \equiv \frac{1}{2} \ln \left(\frac{t+z}{t-z} \right) , \quad (39)$$

¹⁴For these properties to hold true, it is important that the two points are not separated by a time-like interval, which *ipso facto* is guaranteed if they belong to the same locally space-like surface.

and \mathbf{x}_\perp collectively denotes the two co-ordinates perpendicular to the collision axis. In these co-ordinates, the metric tensor

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -\tau^2) \quad (40)$$

has a proper time dependent determinant, $\sqrt{-g} = \tau$. This implies small changes to the equations of motion. The classical Yang-Mills equations (11) become¹⁵

$$\frac{1}{\sqrt{-g}} \mathcal{D}_\alpha (\sqrt{-g} g^{\alpha\beta} g^{\nu\mu} \mathcal{F}_{\beta\mu}) = \delta^{\nu+} \rho_1 + \delta^{\nu-} \rho_2, \quad (41)$$

and the small fluctuation eqs. (14) become

$$\frac{1}{\sqrt{-g}} \mathcal{D}_\alpha (\sqrt{-g} g^{\alpha\beta} g^{\nu\mu} (\mathcal{D}_\beta a_\mu - \mathcal{D}_\mu a_\beta)) - ig g^{\alpha\beta} g^{\nu\mu} \mathcal{F}_{\mu\beta} a_\alpha = 0. \quad (42)$$

In Fock-Schwinger gauge $A^\tau = 0$ and $(\tau, \eta, \mathbf{x}_\perp)$ co-ordinates, these equations can be written out more explicitly to read,

$$\begin{aligned} \mathcal{D}_\eta \partial_\tau a_\eta + \tau^2 \mathcal{D}_i \partial_\tau a_i - \frac{1}{2} \partial_\tau \mathcal{D}_\eta a_\eta - \frac{1}{2} \tau^2 \partial_\tau \mathcal{D}_i a_i &= 0 \\ (\partial_\tau \tau^{-1} \partial_\tau - \tau^{-1} \mathcal{P}_{ii}) a_\eta + \tau^{-1} \mathcal{P}_{i\eta} a_i &= 0 \\ (\partial_\tau \tau \partial_\tau - \tau^{-1} \mathcal{P}_{\eta\eta} - \tau \mathcal{P}_{ii}) a_x + \tau^{-1} \mathcal{P}_{\eta x} a_\eta + \tau \mathcal{P}_{ix} a_i &= 0 \\ (\partial_\tau \tau \partial_\tau - \tau^{-1} \mathcal{P}_{\eta\eta} - \tau \mathcal{P}_{ii}) a_y + \tau^{-1} \mathcal{P}_{\eta y} a_\eta + \tau \mathcal{P}_{iy} a_i &= 0, \end{aligned} \quad (43)$$

where we have introduced the shorthand \mathcal{P} , which is defined as

$$\mathcal{P}_{IJ}^{ab} a_K^b \equiv \mathcal{D}_I^{ac} \mathcal{D}_J^{cb} a_K^b + g f^{acb} \mathcal{F}_{IJ}^c a_K^b. \quad (44)$$

Here the indices I, J, K denote x, y and η , while we use Latin indices i, j to designate the two transverse coordinates x, y . For example, $\partial_i^2 = \partial_x^2 + \partial_y^2 = \partial_\perp^2$. Note that the first equation in eq. (43) is Gauss' law, which can also be written as

$$\mathcal{D}_\eta \partial_\tau a_\eta + \tau^2 \mathcal{D}_i \partial_\tau a_i + ig (\partial_\tau A_\eta) a_\eta + \tau^2 (\partial_\tau A_i) a_i = 0. \quad (45)$$

¹⁵Field equations can be generalized to an arbitrary system of coordinates by trading ordinary derivatives ∂_μ for covariant derivatives ∇_μ (here, *covariant* refers to coordinate transformations, not $SU(3)$ gauge transformations) that involve Christoffel symbols. For the Yang-Mills equations, one should thus write

$$(\nabla_\mu - ig A_\mu) \mathcal{F}^{\mu\nu} = J^\nu, \quad \mathcal{F}^{\mu\nu} \equiv \nabla^\mu A^\nu - \nabla^\nu A^\mu - ig [A^\mu, A^\nu].$$

However, it turns out that for an antisymmetric tensor such as $\mathcal{F}^{\mu\nu}$ one has also (see [64], chapter 5)

$$\nabla_\mu \mathcal{F}^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \mathcal{F}^{\mu\nu}).$$

Moreover, one can show that $\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$, so that the flat-space expression of $\mathcal{F}_{\mu\nu}$ can still be used (provided the two indices are downstairs). In other words, the usual formulas for the field strength should be used only for lower indices, and the metric tensor should be used to raise indices if necessary. For instance, in $A^\tau = A_\tau = 0$ gauge, one has $\mathcal{F}_{\tau\eta} = \partial_\tau A_\eta$ and $\mathcal{F}^{\tau\eta} = g^{\tau\tau} g^{\eta\eta} \mathcal{F}_{\tau\eta} = -\tau^{-2} \partial_\tau A_\eta$, but $\partial^\tau A^\eta = \partial^\tau (-\tau^{-2} A_\eta)$ is not equal to $\mathcal{F}^{\tau\eta}$.

Even though we exclusively work in τ - η coordinates, we shall at times make use of light-cone coordinates defined as

$$x^\pm \equiv \frac{t \pm z}{\sqrt{2}} , \quad (46)$$

with the metric tensor now having the form

$$g_{+-} = g_{-+} = 1 \quad , \quad g_{xx} = g_{yy} = -1 . \quad (47)$$

The transformation from light-cone to τ, η coordinates is given by

$$x^\pm = \frac{\tau e^{\pm\eta}}{\sqrt{2}} , \quad (48)$$

while the Fock-Schwinger gauge condition, expressed in light-cone coordinates, is

$$A^\tau = A_\tau = \frac{1}{\tau} (x^+ A^- + x^- A^+) = 0 . \quad (49)$$

3.3 Inner product for small field fluctuations

Since the equations of motion (42) of the small fluctuations are linear, the set of its solutions is a vector space, and it is sufficient to know a basis of this space in order to be able to construct any solution. For a real background field such as the classical field \mathcal{A}^μ , the evolution in time of the small fluctuations is unitary¹⁶. Therefore, there should be an inner product between pairs of solutions of eq. (42) that remains invariant during the evolution of these solutions. To construct this inner product, rewrite eq. (42) as

$$\mathcal{O}^{\nu\mu} a_\mu = 0 , \quad (50)$$

with

$$\mathcal{O}^{\nu\mu} \equiv D_\alpha \sqrt{-g} (g^{\nu\mu} g^{\alpha\beta} - g^{\nu\beta} g^{\mu\alpha}) D_\beta - ig \sqrt{-g} g^{\nu\alpha} g^{\mu\beta} \mathcal{F}_{\alpha\beta} . \quad (51)$$

Consider now two solutions a_μ and b_μ of eq. (50), and start from the identity

$$0 = \int_\Omega d^4x a_\nu^*(x) \left[\overrightarrow{\mathcal{O}^{\nu\mu}} - \overleftarrow{\mathcal{O}^{\nu\mu}*} \right] b_\mu(x) , \quad (52)$$

where Ω is some domain of space-time. This identity is a trivial consequence of the equations of motion for a^* and b . A remarkable property of the integrand in the right hand side is that it is a total derivative¹⁷,

$$a_\nu^*(x) \left[\overrightarrow{\mathcal{O}^{\nu\mu}} - \overleftarrow{\mathcal{O}^{\nu\mu}*} \right] b_\mu(x) = \partial_\alpha \left[\sqrt{-g} (g^{\nu\mu} g^{\alpha\beta} - \frac{1}{2} g^{\nu\beta} g^{\mu\alpha} - \frac{1}{2} g^{\nu\alpha} g^{\mu\beta}) (a_\nu^*(x) \overleftrightarrow{\mathcal{D}}_\beta b_\mu(x)) \right] . \quad (53)$$

Therefore, one can use Stokes theorem,

$$\int_\Omega d^4x \partial_\alpha F^\alpha = \int_{\partial\Omega} d^3\mathcal{S}_u n_\alpha F^\alpha \quad (54)$$

¹⁶In event of confusion from the apparent structure of the last term of eq. (42), note that the components of the adjoint generators are purely imaginary, and therefore the function that multiplies a^μ in this term is real.

¹⁷For this property to be true, it is crucial that the last term in eq. (51) is real and one should properly take the complex conjugate of the covariant derivatives when they act on the left. This property is in fact closely related to the operator $\mathcal{O}^{\nu\mu}$ being Hermitean; the evolution of the fluctuations is unitary.

where $d^3\mathbf{S}_u$ is the measure on the boundary $\partial\Omega$, and n_α is a normal vector to the boundary, oriented outwards. Let us assume that the boundary $\partial\Omega$ is made of two locally space-like surfaces Σ_1 and Σ_2 , and a third boundary located at infinity in the spatial directions on which all the fields are vanishing. Then eq. (52) is equivalent to

$$\begin{aligned} & \int_{\Sigma_1} d^3\mathbf{S}_u \sqrt{-g} \left(g^{\nu\mu} g^{\alpha\beta} - \frac{1}{2} g^{\nu\beta} g^{\mu\alpha} - \frac{1}{2} g^{\nu\alpha} g^{\mu\beta} \right) n_\alpha \left(a_\nu^*(u) \overleftrightarrow{\mathcal{D}}_\beta b_\mu(u) \right) \\ &= \int_{\Sigma_2} d^3\mathbf{S}_u \sqrt{-g} \left(g^{\nu\mu} g^{\alpha\beta} - \frac{1}{2} g^{\nu\beta} g^{\mu\alpha} - \frac{1}{2} g^{\nu\alpha} g^{\mu\beta} \right) n_\alpha \left(a_\nu^*(u) \overleftrightarrow{\mathcal{D}}_\beta b_\mu(u) \right). \end{aligned} \quad (55)$$

We have thus proved, most generally, that an inner product defined as

$$(a|b) \equiv i \int_{\Sigma} d^3\mathbf{S}_u \sqrt{-g} \left(g^{\nu\mu} g^{\alpha\beta} - \frac{1}{2} g^{\nu\beta} g^{\mu\alpha} - \frac{1}{2} g^{\nu\alpha} g^{\mu\beta} \right) n_\alpha \left(a_\nu^*(u) \overleftrightarrow{\mathcal{D}}_\beta b_\mu(u) \right), \quad (56)$$

is independent of the Cauchy surface Σ used to define it, provided a_μ and b_μ obey the equation of motion of small fluctuations. Note that we have added a factor i to its definition to ensure that it is Hermitean,

$$\begin{aligned} (a|b)^* &= (b|a), \\ (a^*|b^*) &= -(b|a) = -(a|b)^*. \end{aligned} \quad (57)$$

In the special case where Σ is a surface of constant τ and we work in the Fock-Schwinger gauge $A^\tau = 0$, we have $n \cdot \mathcal{D} = \partial_\tau$, and $n \cdot a = 0$, $n \cdot b = 0$. Therefore the inner product simplifies into

$$(a|b) \equiv i \int_{\tau=\text{const}} d^3\mathbf{S}_u \sqrt{-g} g^{\nu\mu} \left(a_\nu^*(u) \overleftrightarrow{\partial}_\tau b_\mu(u) \right). \quad (58)$$

Now let us evaluate the inner product for pairs of field fluctuations taken from the set of the $a_{\pm\mathbf{k}\lambda a}$. Since the inner product does not depend on the chosen time surface and since we know these fields at $x^0 \rightarrow -\infty$ (because they are defined via their initial condition in the remote past), we can evaluate the inner product by using plane wave initial conditions for these fluctuation fields. This gives

$$\begin{aligned} (a_{+\mathbf{k}\lambda a} | a_{-\mathbf{l}\rho b}) &= 0 \\ (a_{+\mathbf{k}\lambda a} | a_{+\mathbf{l}\rho b}) &= \delta_{\lambda\rho} \delta_{ab} (2\pi)^3 2k \delta(\mathbf{k} - \mathbf{l}) \\ (a_{-\mathbf{k}\lambda a} | a_{-\mathbf{l}\rho b}) &= -\delta_{\lambda\rho} \delta_{ab} (2\pi)^3 2k \delta(\mathbf{k} - \mathbf{l}). \end{aligned} \quad (59)$$

Thus this particular basis of the space of solutions of eq. (14) is orthonormal with respect to the invariant inner product defined in eq. (56). Note also that the $a_{+\mathbf{k}\lambda a}$'s represent only one *half of the basis* of the vector space of solutions of eq. (42) –namely the solutions that have a positive frequency in the remote past. The other half is simply obtained by complex conjugation. It is easy to check that any unitary transformation of the positive energy solutions (and a concomitant change to the negative energy ones, that are their complex conjugates) transforms an orthonormal basis into another orthonormal basis, and leaves the formula eq. (38) unchanged. This remark is useful because it leaves us the freedom to label the elements of the basis by other quantities than the Cartesian 3-momentum. This will be true in our specific case where we are interested in a basis in a curvilinear co-ordinate system.

3.3.1 Normalization of the fields and choice of basis

It is important to note that the prefactor in front of the δ functions in eq. (59) exactly cancels the factors that are included in the integration measure in eq. (38), namely one has

$$\int \frac{d^3 \mathbf{k}}{(2\pi)^3 2k} (a_{+\mathbf{k}\lambda a} | a_{+\mathbf{l}\rho b}) = 1 . \quad (60)$$

This remark in fact defines uniquely how the inner product of the basis elements should be normalized given a generic choice for the integration measure in eq. (38); in particular, this rule will be come in handy later when we use other labels than the usual 3-momentum to index the elements of the basis. Moreover, this makes clear that eq. (38) is just one particular representation of the correlator Γ_2 ; there exists such a representation for any orthonormal basis of the space of solutions of eq. (42), as we shall explain now. Thanks to the above inner product, one can spell out a general procedure¹⁸ for constructing the correlator Γ_2 :

- i. Find a complete set of independent *positive energy* solutions a_K of eq. (42), where K denotes collectively (usually a mix of continuous and discrete labels) all the labels necessary to index these solutions.
- ii. This set of solutions should obey the orthogonality condition,

$$(a_K | a_{K'}) = N_K \delta_{KK'} \quad (61)$$

with N_K real and positive definite¹⁹,

- iii. The correlator Γ_2 is then given by

$$\Gamma_2(\mathbf{u}, \mathbf{v}) = \int d\mu_K a_K(\mathbf{u}) a_K^*(\mathbf{v}) , \quad (62)$$

where the measure $d\mu_K$ (a mix of integrals and discrete sums) is such that

$$\int d\mu_K N_K \delta_{KK'} = 1 . \quad (63)$$

It is clear from eqs. (61) and (63) that the Γ_2 given by eq. (62) is independent of how we normalize the solutions (i.e. on the constants N_K), provided we choose the integration measure accordingly. Moreover, we only need to know the form of the solutions at the time of interest, and we can avoid the complication of evolving the plane waves from the past through the forward light-cone. This is particularly helpful when one uses τ, η coordinates, because this system of coordinates has a singularity at $\tau = 0$.

A further simplification is possible because in practice we won't need to use directly eq. (62) for Γ_2 . Indeed, an ensemble of real-valued field fluctuations a^μ that have a 2-point equal-time correlation given by Γ_2 can be generated by the following formula,

$$a^\mu(x) = \int d\mu_K \left[c_K a_K^\mu(x) + c_K^* a_K^{\mu*}(x) \right] , \quad (64)$$

¹⁸In this light, eq. (38) which represents Γ_2 in terms of the $a_{\pm \mathbf{k}\lambda a}$'s, exploits one possible method of constructing such an orthonormal basis. In this case, one starts at $x^0 = -\infty$ with the plane waves, that are known to form an orthonormal basis, and evolves them forward to the time of interest. The time invariance of the inner product then guarantees us that we get an orthonormal basis on the forward light-cone.

¹⁹This means that the solutions a_K will in general be complex solutions.

where the coefficients c_K are random Gaussian-distributed complex numbers whose variance is given by

$$\begin{aligned}\langle c_K c_{K'}^* \rangle &= \frac{N_K}{2} \delta_{KK'} \\ \langle c_K c_{K'} \rangle &= \langle c_K^* c_{K'}^* \rangle = 0.\end{aligned}\tag{65}$$

This method of generating the field fluctuations offers the advantage that it does not require that one diagonalizes the correlation function Γ_2 .

3.4 Free fluctuations in $A^\tau = 0$ gauge

Let us start by calculating the fluctuation correlator in $A^\tau = 0$ gauge and on the surface $\tau = 0^+$ in the *free case*, namely, in the absence of the classical background field. Given the complications introduced by the choice of gauge and the system of curvilinear coordinates, this is a useful exercise to pursue before attacking the more general case of the Glasma background field. In this situation, eqs. (43) simplify into

$$\begin{aligned}\partial_\eta \partial_\tau a_\eta + \tau^2 \partial_i \partial_\tau a_i &= 0 \\ (\partial_\tau \tau^{-1} \partial_\tau - \tau^{-1} \partial_\perp^2) a_\eta + \tau^{-1} \partial_i \partial_\eta a_i &= 0 \\ (\partial_\tau \tau \partial_\tau - \tau^{-1} \partial_\eta^2 - \tau \partial_\perp^2) a_x + \tau^{-1} \partial_\eta \partial_x a_\eta + \tau \partial_i \partial_x a_i &= 0 \\ (\partial_\tau \tau \partial_\tau - \tau^{-1} \partial_\eta^2 - \tau \partial_\perp^2) a_y + \tau^{-1} \partial_\eta \partial_y a_\eta + \tau \partial_i \partial_y a_i &= 0.\end{aligned}\tag{66}$$

3.4.1 Residual gauge freedom

A general solution a^μ to eq. (66) has a priori four components. However, a massless vector field has only two physical degrees of freedom. One of the seemingly independent components of the vector field is removed by the gauge condition $a_\tau = a^\tau = 0$ (this is already implemented in eqs. (66)). However, even after imposing this condition, there is a residual gauge symmetry in the equations of motion, namely, these are invariant under τ independent gauge transformations

$$a^\mu \rightarrow a^\mu + \partial^\mu \Lambda(\eta, \mathbf{x}_\perp),\tag{67}$$

where Λ is an arbitrary τ independent function. As a consequence of this residual gauge freedom, the three remaining components of a^μ are not all physical degrees of freedom.

In order to find the two physical solutions, we begin by finding the unphysical solution. This solution must be a *pure gauge*, which here means it is a τ independent total derivative. As we will see, the unphysical solution is not a dynamical variable but is completely constrained by the initial and boundary conditions. After finding the most general τ independent solution to the equations of motion, the two physical solutions can be determined relatively easily. Their form will be narrowed down by requiring that the three solutions are mutually orthonormal, and then the residual gauge freedom will be fixed by imposing the equations of motion.

In the $(\tau, \eta, \mathbf{x}_\perp)$ system of coordinates, a convenient set of labels for the solutions of eq. (66) is $\nu, \mathbf{k}_\perp, \lambda, a$, where ν is the Fourier conjugate of the space-time rapidity η (as used previously, λ denotes the polarization and a the color). We choose $\lambda = 1, 2$ to denote the physical solutions, and $\lambda = 3$ to be the unphysical one. Since in this section we are in the vacuum, all the colors a

obey exactly the same equations and we can drop this index to keep the notations lighter. With this set of variables, eqs. (61), (62) and (63) take the form

$$\begin{aligned}\Gamma_2(\mathbf{x}, \mathbf{x}') &= \sum_{\lambda=1,2} \int \frac{d^2 \mathbf{k}_\perp d\nu}{(2\pi)^3} a_{\mathbf{k}\lambda}(\tau=0^+, \mathbf{x}) a_{\mathbf{k}\lambda}^*(\tau=0^+, \mathbf{x}') , \\ (a_{\mathbf{k}\lambda} | a_{\mathbf{k}'\lambda'}) &= (2\pi)^3 \delta_{\lambda\lambda'} \underbrace{\delta(\nu - \nu') \delta(\mathbf{k}_\perp - \mathbf{k}'_\perp)}_{\delta(\mathbf{k} - \mathbf{k}')} ,\end{aligned}\tag{68}$$

where we use the shorthands $\mathbf{k} \equiv (\nu, \mathbf{k}_\perp)$ and $\mathbf{x} \equiv (\eta, \mathbf{x}_\perp)$.

3.4.2 Vacuum solutions

In a linear system of coordinates, the $a_{\mathbf{k}\lambda}^\mu(x)$ introduced above would have the simple following parametrization,

$$a_{\mathbf{k}\lambda}^\mu(x) = e^{i\mathbf{k} \cdot \mathbf{x}} e^{-i\omega_{\mathbf{k}} x^0} \varepsilon_{\mathbf{k}\lambda}^\mu ,\tag{69}$$

where $\varepsilon_{\mathbf{k}\lambda}^\mu$ is a constant polarization vector. Note that the minus sign in the exponential that gives the time dependence is necessary in order to ensure that $a_{\mathbf{k}\lambda}^\mu$ is a positive energy solution. But because we work in a curvilinear co-ordinate system, the time dependence of the solutions cannot be a simple exponential. Let us generalize the previous expression by writing

$$a_{\mathbf{k}\lambda}^\mu(\tau, \mathbf{x}) = e^{i\mathbf{k} \cdot \mathbf{x}} \alpha_{\mathbf{k}\lambda}^\mu(\tau) ,\tag{70}$$

where we have combined the polarization vector and the time dependence in a unique quantity that we denote $\alpha_{\mathbf{k}\lambda}^\mu(\tau)$. (Since the equations of motion do not have coefficients that depend explicitly on η or \mathbf{x}_\perp , it is clear that we can still look for solutions whose η and \mathbf{x}_\perp dependence is of the form $\exp(i\mathbf{k} \cdot \mathbf{x})$.) Here again, we will have to make sure that the $a_{\mathbf{k}\lambda}^\mu(\tau, \mathbf{x})$ constructed in this way contains only positive energy contributions.

Let us consider first the unphysical solution. This is a pure gauge solution independent of τ . The most general τ independent solution is of the form

$$\alpha_{\mathbf{k}3}^\mu = \begin{pmatrix} \mathbf{k}_x \\ \mathbf{k}_y \\ \nu \end{pmatrix} \alpha_3(\nu, \mathbf{k}_\perp) ,\tag{71}$$

where $\alpha_3(\nu, \mathbf{k}_\perp)$ is an arbitrary function. The inner product of the unphysical fluctuation $a_{\mathbf{k}3}^\mu$ with one of the physical solutions ($a_{\mathbf{k}'\lambda}^\mu$ with $\lambda = 1, 2$) is

$$\begin{aligned}(a_{\mathbf{k}3} | a_{\mathbf{k}'\lambda}) &= i\tau \alpha_3^*(\nu, \mathbf{k}_\perp) \int_{-\infty}^{+\infty} d\eta \int d^2 \mathbf{x}_\perp e^{i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{x}} \partial_\tau (\mathbf{k}_x \alpha_{\mathbf{k}'\lambda}^x + \mathbf{k}_y \alpha_{\mathbf{k}'\lambda}^y + \nu \tau^{-2} \alpha_{\mathbf{k}'\lambda}^\eta) \\ &= i\tau (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \alpha_3^*(\nu, \mathbf{k}_\perp) \partial_\tau (\mathbf{k}_x \alpha_{\mathbf{k}\lambda}^x + \mathbf{k}_y \alpha_{\mathbf{k}\lambda}^y + \nu \tau^{-2} \alpha_{\mathbf{k}\lambda}^\eta) .\end{aligned}\tag{72}$$

We can satisfy this orthogonality condition by choosing a second solution of the form

$$\alpha_{\mathbf{k}1}^\mu(\tau) = \begin{pmatrix} \mathbf{k}_y \\ -\mathbf{k}_x \\ 0 \end{pmatrix} \alpha_1(\tau, \nu, \mathbf{k}_\perp) .\tag{73}$$

The functional form of α_1 should be fixed by the equations of motion. Substituting the above expression into the equations of motion (66) yields a differential equation for α_1 , whose general solution can be expressed in terms of the Hankel functions $H_{i\nu}^{(1)}$ and $H_{i\nu}^{(2)}$,

$$\alpha_1(\tau, \nu, \mathbf{k}_\perp) = a_{\mathbf{k}} H_{i\nu}^{(1)}(k_\perp \tau) + b_{\mathbf{k}} H_{i\nu}^{(2)}(k_\perp \tau) . \quad (74)$$

Recall here the integral representation of the Hankel functions,

$$\begin{aligned} H_{i\nu}^{(1)}(x) &= -\frac{i}{\pi} e^{+\pi\nu/2} \int_{-\infty}^{+\infty} e^{ix \cosh t + i\nu t} dt \\ H_{i\nu}^{(2)}(x) &= +\frac{i}{\pi} e^{-\pi\nu/2} \int_{-\infty}^{+\infty} e^{-ix \cosh t - i\nu t} dt , \end{aligned} \quad (75)$$

The convention set by eq. (69) implies that only $H_{i\nu}^{(2)}(k_\perp \tau)$ has the appropriate frequency sign to be one the $a_{\mathbf{k}\lambda}$'s. We can therefore set $a_{\mathbf{k}} = 0$ and keep only the second term in eq. (74). The value of $b_{\mathbf{k}}$ can then be determined by the orthogonality condition. For this, we need the identity

$$H_{i\nu}^{(2)*}(x) \overset{\leftrightarrow}{\partial}_x H_{i\nu}^{(2)}(x) = -\frac{4ie^{-\pi\nu}}{\pi x} , \quad (76)$$

from which we obtain

$$(a_{\mathbf{k}1} | a_{\mathbf{k}'1}) = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') |b_{\mathbf{k}}|^2 \mathbf{k}_\perp^2 \frac{4e^{-\pi\nu}}{\pi} . \quad (77)$$

In order to get the same normalization as in eq. (68), we obtain (up to an irrelevant phase)

$$b_{\mathbf{k}} = \frac{\sqrt{\pi} e^{\pi\nu/2}}{2\mathbf{k}_\perp} , \quad (78)$$

and the first physical solution reads

$$a_{\mathbf{k}1}^\mu(\mathbf{x}) = \frac{\sqrt{\pi} e^{\pi\nu/2}}{2\mathbf{k}_\perp} \begin{pmatrix} \mathbf{k}_y \\ -\mathbf{k}_x \\ 0 \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{x}} H_{i\nu}^{(2)}(\mathbf{k}_\perp \tau) . \quad (79)$$

The second physical solution can be found by requiring that it be orthogonal to the two solutions we have so far. This restricts it to be of the form

$$\alpha_{\mathbf{k}2}^\mu = \begin{pmatrix} \nu \mathbf{k}_x \alpha_{2\perp}(\tau, \nu, \mathbf{k}_\perp) \\ \nu \mathbf{k}_y \alpha_{2\perp}(\tau, \nu, \mathbf{k}_\perp) \\ -\alpha_{2\eta}(\tau, \nu, \mathbf{k}_\perp) \end{pmatrix} , \quad (80)$$

provided that $k_\perp^2 \tau^2 \partial_\tau \alpha_{2\perp} = \partial_\tau \alpha_{2\eta}$. This last constraint was derived by substituting the general form of the second solution eq. (80) into the orthogonality condition eq. (72) with the unphysical solution. It turns out that this is the same constraint needed for this solution to fulfill Gauss's law.

Dynamical equations for the functions $\alpha_{2\perp}$ and $\alpha_{2\eta}$ can be found by substituting eq. (80) into eqs. (66). One obtains,

$$\begin{aligned} \partial_\tau^3 \alpha_{2\eta} - \frac{1}{\tau} \partial_\tau^2 \alpha_{2\eta} + \left(\frac{\nu^2 + 1}{\tau^2} + \mathbf{k}_\perp^2 \right) \partial_\tau \alpha_{2\eta} &= 0 , \\ \partial_\tau^3 \alpha_{2\perp} + \frac{3}{\tau} \partial_\tau^2 \alpha_{2\perp} + \left(\frac{\nu^2 + 1}{\tau^2} + \mathbf{k}_\perp^2 \right) \partial_\tau \alpha_{2\perp} &= 0 . \end{aligned} \quad (81)$$

The positive energy solutions to the above third order differential equations can be written as

$$\begin{aligned}\alpha_{2\eta} &= \text{const} \times \int^\tau d\tau' \tau' H_{i\nu}^{(2)}(\mathbf{k}_\perp \tau') \\ \alpha_{2\perp} &= \text{const} \times \int^\tau \frac{d\tau'}{\tau'} H_{i\nu}^{(2)}(\mathbf{k}_\perp \tau') .\end{aligned}\tag{82}$$

Note that the differential equations above imply that the functional form of $\alpha_{2\perp}$ and $\alpha_{2\eta}$ are only determined up to an arbitrary τ independent function. This corresponds to a residual gauge freedom in which we can always add to the second physical solution eq. (80) a pure gauge solution having the form of eq. (71). This residual gauge freedom can be removed by imposing an additional gauge fixing condition. For example, in simulations of classical Yang–Mills one typically imposes transverse Coulomb gauge.

The integrals over the Hankel functions can be written in terms of hypergeometric functions but their form is not very enlightening. To streamline our notation, following Makhlin [58], let us define

$$R_{b,\alpha}^{(a)}(\mathbf{k}_\perp \tau) \equiv \int^\tau dx x^b H_\alpha^{(a)}(\mathbf{k}_\perp x) .\tag{83}$$

The properly normalized form for the second physical degree of freedom is

$$a_{\mathbf{k}2}^\mu(\tau, \eta, \mathbf{x}_\perp) = \frac{\sqrt{\pi} e^{\pi\nu/2}}{2\mathbf{k}_\perp} \begin{pmatrix} \nu \mathbf{k}_x R_{-1,i\nu}^{(2)}(\mathbf{k}_\perp \tau) \\ \nu \mathbf{k}_y R_{-1,i\nu}^{(2)}(\mathbf{k}_\perp \tau) \\ -R_{+1,i\nu}^{(2)}(\mathbf{k}_\perp \tau) \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{x}} .\tag{84}$$

For $\mathbf{k}_\perp \tau \ll 1$, we can make use of the series expansion of the Hankel functions and rewrite the solution as

$$a_{\mathbf{k}2}^\mu(\mathbf{x}) \approx \frac{\sqrt{\pi} e^{\pi\nu/2}}{2\mathbf{k}_\perp} \begin{pmatrix} \mathbf{k}_x \\ \mathbf{k}_y \\ -(\mathbf{k}_\perp \tau)^2/(\nu + 2i) \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{x}} H_{i\nu}^{(2)}(\mathbf{k}_\perp \tau) .\tag{85}$$

From these explicit solutions, we will construct in appendix B the correlation function Γ_2 for free fields on the initial surface $\tau = 0^+$. However, for the purposes of generating a Gaussian ensemble of fluctuations with the proper variance, the above results along with eqs. (64) and (65) are sufficient.

4 Small fluctuations in the Glasma

After our extended discussion of the free fluctuations in $A^\tau = 0$ gauge, we now turn to the derivation of the small fluctuations spectrum in the Glasma. We shall first write down the small fluctuation equations of motion in the presence of the background classical fields. The main difficulty here is that the Glasma background fields are not known analytically at arbitrary proper times. They are however known in closed form at $\tau = 0^+$ in terms of the classical CGC fields before the collision. We will perform a small time expansion, valid at very short proper times $\tau \ll Q_s^{-1}$, of both the classical fields and the small fluctuation fields and show that the fluctuations only depend on the classical gauge fields immediately after the collision. As our final result, we will obtain explicit expressions for an orthonormal basis of small fluctuations that generalize eqs. (79) and (84) to the case of a non-zero background field at small proper times.

4.1 Structure of the Glasma background field

In the Fock-Schwinger gauge, the classical gauge field configurations can be expressed as [47,51]

$$\begin{aligned} A^i &= \theta(-x^+)\theta(x^-)\alpha_1^i + \theta(x^+)\theta(-x^-)\alpha_2^i + \theta(x^+)\theta(x^-)\mathcal{A}^i \\ A^\eta &= \theta(x^+)\theta(x^-)\mathcal{A}^\eta \end{aligned} \quad (86)$$

The fields $\alpha_{1,2}^i$ are the color fields of the two nuclei before the collision, that take the form of transverse pure gauge fields, while \mathcal{A}^μ denotes the classical fields after the collision.

Since we are interested in the spectrum of fluctuations at $\tau = 0^+$ we need only the behavior of the background fields shortly after the collision. The classical Glasma fields in the forward light cone can be expanded, in all generality, at early times as

$$\mathcal{A}_I = \sum_{n=0}^{\infty} \mathcal{A}_{(n)I} \tau^n \quad (87)$$

The initial conditions for these background fields at $\tau = 0^+$ are obtained by matching the Yang-Mills equations just below and just above the forward light-cone (to ensure a regular behavior of the field equations). One obtains [47] for the fields and their time derivatives,

$$\begin{aligned} \mathcal{A}^i(\tau = 0^+) &= \alpha_1^i + \alpha_2^i \\ \mathcal{A}_\eta(\tau = 0^+) &= 0 \\ \mathcal{E}_i(\tau = 0^+) &= \tau \partial_\tau \mathcal{A}_i|_{\tau=0} = 0 \\ \mathcal{E}^\eta(\tau = 0^+) &= \frac{1}{\tau} \partial_\tau \mathcal{A}_\eta|_{\tau=0} = ig [\alpha_1^i, \alpha_2^i] . \end{aligned} \quad (88)$$

As alluded to previously, explicit analytical solutions are known for the fields $\alpha_{1,2}^i$. The Taylor expansions of eqs. (87) begin with

$$\mathcal{A}_i = \mathcal{A}_i(0^+) + \mathcal{O}(\tau) \quad , \quad \mathcal{A}_\eta = \frac{1}{2} \mathcal{E}^\eta(0^+) \tau^2 + \mathcal{O}(\tau^3) . \quad (89)$$

At this point it is useful to introduce some extra notation. Based on the Taylor expansion in eq. (87) of the classical background field, it will be convenient to introduce an analogous expansion of the covariant derivatives²⁰ and projectors as defined in eq. (44),

$$\begin{aligned} \mathcal{D}_\mu &= \sum_n \tau^n \mathcal{D}_{(n)\mu} , \\ \mathcal{P}_{\mu\nu} &= \sum_n \tau^n \mathcal{P}_{(n)\mu\nu} . \end{aligned} \quad (90)$$

Naturally, the coefficients $\mathcal{D}_{(n)\mu}$ and $\mathcal{P}_{(n)\mu\nu}$ can be written in terms of the Taylor coefficients of the classical background field, $\mathcal{A}_{(n)I}$. For example, if we perform the Taylor expansion of the covariant derivative, $\mathcal{D}_{(n)\mu}$, the leading coefficients can be expressed in terms of the Taylor coefficients of the classical background field,

$$\mathcal{D}_{(0)i}^{ab} = \delta^{ab} \partial_i - ig \mathcal{A}_{(0)i}^{ab} \quad , \quad \mathcal{D}_{(0)\eta}^{ab} = \delta^{ab} \partial_\eta \quad , \quad \mathcal{D}_{(2)\eta}^{ab} = -ig \mathcal{A}_{(2)\eta}^{ab} . \quad (91)$$

²⁰Of course, the D_τ derivative does not need to be expanded, since in the Fock-Schwinger gauge one has $D_\tau \equiv \partial_\tau$.

In writing down the above expressions we have used the fact that the transverse components of the background field have non-vanishing zero order Taylor coefficients in contrast to the longitudinal component of the background field whose leading behavior starts with τ^2 . Similarly we can express the Taylor coefficients of $\mathcal{P}_{(n)\mu\nu}$ in terms of the Taylor coefficients of the classical background field. The terms which will be needed later on in our discussion include,

$$\begin{aligned}\mathcal{P}_{(0)\eta i}^{ab} &= \mathcal{P}_{(0)i\eta}^{ab} = \mathcal{D}_{(0)i}^{ab} \partial_\eta, \\ \mathcal{P}_{(0)ij}^{ab} &= \delta^{ab} \partial_i \partial_j - ig \mathcal{A}_{(0)i}^{ab} \partial_j - ig \mathcal{A}_{(0)j}^{ab} \partial_i - ig (\partial_i \mathcal{A}_{(0)j})^{ab} - g^2 \mathcal{A}_{(0)i}^{ac} \mathcal{A}_{(0)j}^{cb}, \\ \mathcal{P}_{(2)\eta\eta}^{ab} &= -2ig \mathcal{A}_{(2)\eta}^{ab} \partial_\eta.\end{aligned}\tag{92}$$

In deriving the above expressions we have used the fact that the background field is boost invariant, $\partial_\eta \mathcal{A}_I \equiv 0$. It is this property of the background field that leads to the simple form of $\mathcal{P}_{(0)\eta i}$ and yields the factor of two in the expression²¹ for $\mathcal{P}_{(2)\eta\eta}$.

4.2 Early time behavior

In the case of the $\tau, \eta, \mathbf{x}_\perp$ system of coordinates and in the Fock–Schwinger gauge, the equations of motion for the fluctuations propagating over such a background field are written explicitly in eqs. (43). Solving for the full time dependence of the fluctuations is both intractable (since the time dependence of the background field is only known numerically) and unnecessary (since we only need the spectrum of fluctuations at early times).

A crucial property of the background Glasma fields is that they are invariant under boosts in the longitudinal direction. This implies that the fields \mathcal{A}_i and \mathcal{A}_η in the forward light-cone, after the collision, are independent of the space-time rapidity η . Thus the variable ν , the Fourier conjugate of η , is a conserved quantum number for fluctuations propagating over the Glasma fields, that can be used to label the elements of the basis. We can therefore write the elements of the basis as

$$a_{\nu\lambda}^\mu(\tau, \eta, \mathbf{x}_\perp) \equiv e^{i\nu\eta} \beta_{\nu\lambda}^\mu(\tau, \mathbf{x}_\perp),\tag{93}$$

where $\lambda = 1, 2$ is a polarization index and \mathbf{l} collectively represents the remaining quantum numbers necessary to label the basis. The main difference with the free case (see eq. (70)) is that we cannot assume that the \mathbf{x}_\perp dependence of the fluctuations has the form of a plane wave and instead represent it more generally as the function $\beta_{\nu\lambda}^\mu(\tau, \mathbf{x}_\perp)$. This is because the background field has a non-trivial dependence on \mathbf{x}_\perp . A further consequence is that, in contrast to the vacuum case, the remaining quantum numbers encoded in \mathbf{l} will not simply be transverse momenta. As in the free case, we shall keep only solutions that have positive frequencies in this basis.

We would now like to motivate one of the main results of this work; a modified form of the linearized equation of motion eq. (43), which captures the early time behavior of a quantum fluctuation propagating on top of the classical background field. The simplest way to arrive at our result is to simply replace all of the projectors appearing in eq. (43) with the corresponding zeroth order Taylor coefficient from the proper time expansion given in eq. (90). Following this procedure gives essentially the right result except for one subtlety. We will argue that we also need to include

²¹It may be instructive to work this term out explicitly. Since the field strength tensor is anti-symmetric we have $\mathcal{P}_{\eta\eta}^{ab} = \mathcal{D}_{\eta}^{ac} \mathcal{D}_{\eta}^{cb}$. We can therefore write $\mathcal{P}_{(2)\eta\eta}^{ab} = \mathcal{D}_{(0)\eta}^{ac} \mathcal{D}_{(2)\eta}^{cb} + \mathcal{D}_{(2)\eta}^{ac} \mathcal{D}_{(0)\eta}^{cb}$. From eq. (91) we know that $\mathcal{D}_{(0)\eta}^{ab} = \delta^{ab} \partial_\eta$ and therefore commutes with the boost invariant background field. The final result is then $\mathcal{P}_{(2)\eta\eta}^{ab} = 2\mathcal{D}_{(2)\eta}^{ab} \partial_\eta$.

one term, $\tau^2 \mathcal{P}_{(2)\eta\eta}$, that appears at higher order in the expansion of the projector. The resulting small-time linearized equations of motion are,

$$\begin{aligned} (\partial_\tau \tau^{-1} \partial_\tau - \tau^{-1} \mathcal{P}_{(0)ii}) a_\eta + \tau^{-1} \mathcal{P}_{(0)i\eta} a_i &= 0, \\ (\partial_\tau \tau \partial_\tau - \tau^{-1} \mathcal{P}_{(0)\eta\eta} - \tau \mathcal{P}_{(2)\eta\eta} - \tau \mathcal{P}_{(0)ii}) a_x + \tau^{-1} \mathcal{P}_{(0)\eta x} a_\eta + \tau \mathcal{P}_{(0)ix} a_i &= 0, \\ (\partial_\tau \tau \partial_\tau - \tau^{-1} \mathcal{P}_{(0)\eta\eta} - \tau \mathcal{P}_{(2)\eta\eta} - \tau \mathcal{P}_{(0)ii}) a_y + \tau^{-1} \mathcal{P}_{(0)\eta y} a_\eta + \tau \mathcal{P}_{(0)iy} a_i &= 0. \end{aligned} \quad (94)$$

The necessity of including the term $(\mathcal{P}_{(2)\eta\eta})$ can be seen by looking at the structure of the operator acting on a_x in the second equation (or equivalently acting on a_y in the third equation) above;

$$(\partial_\tau \tau \partial_\tau - \tau^{-1} \mathcal{P}_{(0)\eta\eta} - \tau \mathcal{P}_{(2)\eta\eta} - \tau \mathcal{P}_{(0)ii}) a_x, \quad (95)$$

By examining the power counting in τ of each term in this operator, it would be inconsistent to ignore the τ^2 component of $\mathcal{P}_{\eta\eta}$ which is of the same order as $\mathcal{P}_{(0)ii}$. One could argue that both $\mathcal{P}_{(0)ii}$ and $\mathcal{P}_{(2)\eta\eta}$ are suppressed by τ^2 at early times relative to $\mathcal{P}_{(0)\eta\eta}$ and therefore can both be ignored. If we drop these terms, when we *turn off* the background field, we would not recover the vacuum wavefunctions and this is clearly unsatisfactory. We therefore conclude that we need to include all projectors that are of the same order in τ as the constants appearing in the vacuum case. By the argument presented we should also include the term $\mathcal{P}_{(2)\eta\eta}$ in order to have the correct power counting.

A more formal way of arriving at eq. (94) is by considering the series expansion of the small fluctuations, a_I . Using the method of Frobenius one finds that the leading τ behavior of the transverse components behaves as $a_i \sim \tau^{i\nu}$ while that of the longitudinal component goes as $a_\eta \sim \tau^{2+i\nu}$. These coefficients are exactly those needed to reproduce the small τ expansion of the physical solutions found in the vacuum case. Furthermore, if we continue to use the Frobenius method we find that only a small subset of the Taylor Coefficients in eq. (90) are needed to determine the lowest order coefficient in the Frobenius expansion of a_I . These are precisely the Taylor coefficients that have been included in eq. (94).

Finally, we need to discuss why we have not included Gauss's law in eq. (94). The reasoning is that Gauss's law is not a dynamical equation but a constraint, and therefore not amenable to a series expansion. This can be seen by noting that Gauss's law,

$$\mathcal{G} \equiv \tau^{-1} \mathcal{D}_\eta \mathcal{E}^\eta + \tau \mathcal{D}_i \mathcal{E}^i = 0 \quad (96)$$

is a constant of motion ($\partial_\tau \mathcal{G} = 0$) and therefore if Gauss's law is obeyed at $\tau = 0^+$ it will remain satisfied for all times.

4.3 First physical solution

For the first physical solution, we take $a_{\nu I 1}^\eta = 0$ as was done in the vacuum case. With this choice, the first equation in (94) coincides with Gauss's law. The last two equations control the evolution of the transverse components $a_{\nu I 1}^i$. Since we expect the time dependence in eq. (93) to enter in the same way as in the vacuum case, we postulate that the solution at early times will be of the form

$$a_{\nu I 1}^\mu(\tau, \eta, \mathbf{x}_\perp) = \frac{\sqrt{\pi} e^{\pi\nu/2}}{2Q_{\nu I 1}} \begin{pmatrix} b_{\nu I 1}^x(\mathbf{x}_\perp) \\ b_{\nu I 1}^y(\mathbf{x}_\perp) \\ 0 \end{pmatrix} e^{i\nu\eta} H_{i\nu}^{(2)}(Q_{\nu I 1} \tau). \quad (97)$$

Next we substitute eq. (97) into the early time linearized equation of motion (94). Requiring that eq. (97) is a solution to the equations of motion leads to,

$$\begin{aligned} - \left[\mathcal{D}_{(0)y} \mathcal{D}_{(0)y} + \mathcal{P}_{(2)\eta\eta}^{(\nu)} \right] b_{\nu l1}^x + \mathcal{P}_{(0)yx} b_{\nu l1}^y &= Q_{\nu l1}^2 b_{\nu l1}^x \\ - \left[\mathcal{D}_{(0)x} \mathcal{D}_{(0)x} + \mathcal{P}_{(2)\eta\eta}^{(\nu)} \right] b_{\nu l1}^y + \mathcal{P}_{(0)xy} b_{\nu l1}^x &= Q_{\nu l1}^2 b_{\nu l1}^y . \end{aligned} \quad (98)$$

We have explicitly included a superscript (ν) on $\mathcal{P}_{(2)\eta\eta}^{(\nu)}$ in order to remind ourselves that the derivative with respect to η should be replaced by $i\nu$ when acting on the exponential in eq. (97). We should also stress that the equations above only depend on the background fields at $\tau = 0^+$, which are known analytically.

Solving eqs. (98) amounts to finding the eigenvalues $Q_{\nu l1}^2$ and eigenfunctions (the doublets $(b_{\nu l1}^x(\mathbf{x}_\perp), b_{\nu l1}^y(\mathbf{x}_\perp))$) of an Hermitean operator. Since this operator is Hermitean, its spectrum is made of real eigenvalues, and its eigenfunctions can be chosen to form an orthonormal basis,

$$\int d^2 \mathbf{x}_\perp b_{\nu l1}^{i*}(\mathbf{x}_\perp) b_{\nu' l' 1}^i(\mathbf{x}_\perp) = \delta_{ll'} . \quad (99)$$

Note that the choice of normalization in eqs. (97) and (99) is such that the inner product defined in eq. (68) is satisfied

$$(a_{\nu l1} | a_{\nu' l' 1}) = 2\pi \delta(\nu - \nu') \delta_{ll'} . \quad (100)$$

The operator that we need to diagonalize in eqs. (98) has a spectrum that is twice larger than the size expected for the space of solutions with polarization $\lambda = 1$. Half of this spectrum is incompatible with Gauss' law and must be discarded²².

4.4 Second physical solution

As was the case for the first physical solution, the second physical solution will maintain the same τ dependence at $\tau = 0^+$ but with a modified dispersion relation. We therefore write the most general form of the second physical solution as

$$a_{\nu l2}^\mu(\tau, \eta, \mathbf{x}_\perp) = \frac{\sqrt{\pi} e^{\pi\nu/2}}{2Q_{\nu l2}} \begin{pmatrix} b_{\nu l2}^x(\mathbf{x}_\perp) R_{-1, i\nu}^{(2)}(Q_{\nu l2} \tau) \\ b_{\nu l2}^y(\mathbf{x}_\perp) R_{-1, i\nu}^{(2)}(Q_{\nu l2} \tau) \\ b_{\nu l2}^\eta(\mathbf{x}_\perp) R_{+1, i\nu}^{(2)}(Q_{\nu l2} \tau) \end{pmatrix} e^{i\nu\eta} . \quad (101)$$

Following the same procedure as for the first solution, we substitute eq. (101) into the linearized equations of motion given by eq. (94). Requiring that eq. (101) is a solution to the equation of motion at lowest order in τ leads to the following equations for $b_{\nu l2}^i$

$$\begin{aligned} b_{\nu l2}^x &= i\nu \mathcal{D}_{(0)x} b_{\nu l2}^\eta \\ b_{\nu l2}^y &= i\nu \mathcal{D}_{(0)y} b_{\nu l2}^\eta \\ -\mathcal{P}_{(0)ii} b_{\nu l2}^\eta(\mathbf{x}_\perp) &= Q_{\nu l2}^2 b_{\nu l2}^\eta(\mathbf{x}_\perp) . \end{aligned} \quad (102)$$

²²In the free case, this operator is $\mathcal{O}_{ij} \equiv -\partial_\perp^2 \delta_{ij} + \partial_i \partial_j$. It has two types of eigenfunctions: (i) $b^i = \partial_i \chi$, with eigenvalue $Q^2 = 0$, and (ii) $b^i = \epsilon_{ij} \partial_j \chi$, with eigenvalue $Q^2 = k_\perp^2$ (where $\chi(\mathbf{x}_\perp) \equiv \exp(i\mathbf{k}_\perp \cdot \mathbf{x}_\perp)$). Only the second eigenfunction is compatible with Gauss' law $\partial_i b^i = 0$. Interestingly, one of the perturbative solutions of the classical equations of motion at large transverse momenta in the forward light-cone has an identical structure [47].

The first two of these equations simply give $b_{\nu l2}^{x,y}$ in terms of $b_{\nu l2}^\eta$. The third equation determines $b_{\nu l2}^\eta$ as an eigenfunction of the operator $-\mathcal{P}_{(0)ii}$, with eigenvalue $Q_{\nu l2}^2$. Because this operator is Hermitean, these eigenvalues are real, and the eigenfunctions are mutually orthogonal,

$$\int d^2 \mathbf{x}_\perp b_{\nu l2}^{\eta*}(\mathbf{x}_\perp) b_{\nu l'2}^\eta(\mathbf{x}_\perp) = \delta_{ll'}. \quad (103)$$

We can therefore write the final form of the second physical solution in terms of the single eigenfunction $b_{\nu l2}^\eta$

$$a_{\nu l2}^\mu(\tau, \eta, \mathbf{x}_\perp) = \frac{\sqrt{\pi} e^{\pi\nu/2}}{2Q_{\nu l2}} \begin{pmatrix} i\nu R_{-1,i\nu}^{(2)}(Q_{\nu l2}\tau) \mathcal{D}_{(0)x} \\ i\nu R_{-1,i\nu}^{(2)}(Q_{\nu l2}\tau) \mathcal{D}_{(0)y} \\ R_{+1,i\nu}^{(2)}(Q_{\nu l2}\tau) \end{pmatrix} b_{\nu l2}^\eta(x_\perp) e^{i\nu\eta}, \quad (104)$$

where $b_{\nu l2}^\eta(x_\perp)$ is a solution to the eigenvalue equation (102). Finally, let us rewrite the solution using the small time approximation of $R_{\pm 1,i\nu}^{(2)}$, as done in the vacuum case (see eq. (85))

$$a_{\mathbf{k}2}^\mu(\mathbf{x}) \approx \frac{\sqrt{\pi} e^{\pi\nu/2}}{2Q_{\nu l2}} \begin{pmatrix} \mathcal{D}_{(0)x} \\ \mathcal{D}_{(0)y} \\ -(Q_{\nu l2}\tau)^2/(\nu + 2i) \end{pmatrix} b_{\nu l2}^\eta(x_\perp) e^{i\nu\eta} H_{i\nu}^{(2)}(Q_{\nu l2}\tau). \quad (105)$$

5 Outline of an algorithm for numerical computations

The results of the previous section provide all the ingredients we need in order to evaluate an inclusive quantity such as the energy-momentum tensor, resumming in the calculation both the large logs of $1/x_{1,2}$ and the secular terms that plague fixed order calculations. The algorithm for performing such a calculation can be broken down into several independent steps:

- i. Solve the JIMWLK equation.
 - i.a Generate an ensemble of color source densities $\rho_a(\mathbf{x}_\perp)$ (or, equivalently, of Wilson lines $\Omega_{ab}(\mathbf{x}_\perp)$) that represent the distribution $W[\rho]$ at large x , close to the fragmentation region of a nucleus.
 - i.b For each of these configurations, evolve it to smaller x by using the Langevin formulation of the JIMWLK equation. This amounts to performing a random walk on the space of mappings from \mathbb{R}^2 to the group $SU(3)$ [65,66].
- ii. Pick two elements (one for each nucleus) in the above ensembles for each projectile, evolved at the values of x relevant to the observable of interest. Compute the gauge fields and their first time derivatives on the initial surface $\tau = 0^+$ immediately after the collision [52,56].
- iii. Generate fluctuations on top of the classical Glasma fields.
 - iii.a Solve the eigenvalue equations in eqs. (98) and (102). In the former case, only the solutions that fulfill Gauss's law should be kept. It should be noted that in a lattice discretization, this amounts to diagonalizing large but sparse matrices.

- iii.b Evaluate the Hankel functions $H_{i\nu}^{(2)}$ and the hypergeometric functions $R_{\pm 1, i\nu}^{(2)}$ at the initial time of interest. (Note that because they oscillate as $\ln(\tau) \rightarrow -\infty$, this initial time cannot be exactly zero.) To do this, one can go back to their defining differential equations, and solve them numerically, keeping only the solution that becomes a positive frequency plane wave as $\tau \rightarrow +\infty$.
- iii.c Superimpose on top of the classical Glasma fields (obtained in step ii), a linear combination of the small fluctuations that are obtained by solving the eigenvalue equations in eqs. (97) and (104), multiplied by random Gaussian coefficients that have a variance²³ given by eq. (65).
- iv. For each initial condition generated in this way, solve numerically the classical Yang-Mills equations forward in time, up to the proper time at which the observable should be evaluated. Repeat steps iii and iv in order to do a Monte-Carlo evaluation of the average over the fluctuations of the initial gauge fields.

6 Applications and Outlook

With this work, we have completed the resummation of all the leading contributions from quantum fluctuations to inclusive quantities at early times, in the Color Glass Condensate effective field theory approach to high energy nucleus-nucleus collisions. These quantum fluctuations can be factorized into $\nu = 0$ and $\nu \neq 0$ modes, where ν is the Fourier conjugate to the space-time rapidity η . The expression in eq. (5) described the contribution of $\nu = 0$ modes, which correspond to summing the leading logarithmic $\alpha_s \ln(1/x_{1,2})$ contributions. Our final expression for the energy-momentum tensor, extending the expression in eq. (5) to include the leading secular terms, is

$$\begin{aligned} \langle T^{\mu\nu} \rangle_{\text{LLx+LInst.}} &= \int [D\rho_1 D\rho_2] W_{x_1}[\rho_1] W_{x_2}[\rho_2] \\ &\times \int [\mathcal{D}\alpha] F_0[\alpha] T_{\text{LO}}^{\mu\nu}[\mathbb{A}[\rho_1, \rho_2] + \alpha](x), \end{aligned} \quad (106)$$

where the weight functionals $W_{x_{1,2}}[\rho_{1,2}]$ satisfy the JIMWLK renormalization group equation in eq. (6). The argument $\mathbb{A} \equiv (\mathcal{A}, \mathcal{E})$ denotes collectively the components of the classical fields and their canonically conjugate momenta on the initial proper time surface. These quantities are functionals of ρ_1, ρ_2 and, as discussed previously, analytical expressions for these are available at $\tau = 0^+$ [47,51,55,56]. The initial spectrum of fluctuations $F_0[\alpha]$, defined in eq. (28), requires that one compute small fluctuations around the classical background fields \mathbb{A} as $\tau \rightarrow 0^+$. The formal expressions for these were derived in section 4. In section 5, we have outlined a practical algorithm to compute the path integral over small fluctuations. As numerical algorithms for solving the JIMWLK equation are now also available [65] and have been successfully implemented [66], a full fledged numerical computation of eq. (106) is feasible in the near future. We should emphasize that this equation is valid for any inclusive quantity, not just the energy-momentum tensor.

There are several applications of this formalism to understand key features of early time dynamics in heavy ion collisions. We shall discuss a few of these here.

²³Note that the (arbitrary) factor N_K in this variance is cancelled by the normalization of the eigenfunctions a_K and of the integration measure $d\mu_K$. Indeed, in eq. (64) one has $c_K \sim \sqrt{N_K}$, $a_K \sim \sqrt{N_K}$ and $d\mu_K \sim 1/N_K$.

- *Early thermalization* It is important to emphasize that a numerical simulation of eq. (106) would describe the real time evolution of a quantum field theory that goes far beyond purely classical contributions but includes as well important quantum effects to all orders in perturbation theory. (For similar considerations of the relative roles of classical versus quantum effects within the 2PI framework, we refer readers to refs. [67–69] and references therein.) In ref. [1], we developed the corresponding formalism for a scalar ϕ^4 theory, and demonstrated that the system developed an equation of state, allowing one to write the energy-momentum conservation equation for the resummed energy momentum tensor $T^{\mu\nu}$ as the closed form set of equations corresponding to the ideal hydrodynamics satisfied by a perfect relativistic fluid. We interpreted this result as arising from a phase decoherence of the classical trajectories in eq. (106), with the different initial conditions given by the spectrum of initial quantum fluctuations. As discussed previously, we expect the same to occur in QCD as well.

It is widely believed that the strong hydrodynamic behavior seen in heavy ion collisions requires early thermalization. However, as shown for the scalar theory, and may likely also be true for gauge theories, this is not a necessary condition for (nearly) perfect fluidity. It is interesting to ask what is the proper condition for thermalization in a quantum field theory. One such criterion is based on Berry’s conjecture [40,41,43,44], that states that in a quantum system whose classical counterpart is chaotic, high lying energy eigenfunctions look like Gaussian random functions. It was later argued by Srednicki [42] that such eigenstates would appear to be thermal, for example, if one measures the single particle distribution. Since the underlying classical Yang-Mills theory is believed to be chaotic [45,46], the quantum system described by eq. (106) is a good candidate to explore these ideas in a quantum field theory. In our approach, the initial state is not an energy eigenstate, but rather a coherent state. It is formulated as a sum of plane waves with random Gaussian coefficients (see eqs. (64) and (65)) but there is no constraint that restrict the Gaussian random eigenstates to states of a given energy, in contrast to the states postulated by Berry. The self-interactions of the fields lead to a loss of this initial coherence, and it would be very interesting to see if thermalization occurs on the same time-scales as decoherence.

Alternatively, one can look at the spectral function, defined in terms of the imaginary part of the retarded Green function, for the appearance of quasi-particle behavior, which allows for a kinetic theory description in terms of single particle distributions. It is interesting to explore whether there is a region of overlap between a description in terms of high occupation number fields and a kinetic theory description in terms of quasi-particles [70,71]. Such a regime may equivalently be described by a coupled set of equations for the classical fields and quasi-particle modes [72] which is reminiscent of the description of superfluids in condensed matter physics. Indeed, it is conceivable that the best description of such an overpopulated system, for intermediate times, might be as a Bose-Einstein superfluid [73], before the inelastic processes in the Glasma begin to dominate [74,75] and lead eventually to a conventional kinetic description.

While the mechanism for thermalization is non-perturbative, it is still unclear whether the mechanism is weak coupling or strong coupling. One might anticipate that the former is more likely at higher energies due to the increasing dominance of semi-hard scales, but where such a transition occurs is unknown. It is intriguing that while the mechanism of thermalization in AdS/CFT inspired strong coupling approaches appears very different (more “top down” than “bottom up”), there appear to be technical similarities between aspects of our weak

coupling approach and these approaches [76,77].

- *Sphaleron transitions*

As emphasized, our master formula in eq. (106) is valid not only for the expectation value of the energy-momentum tensor but also for other inclusive quantities. One such example is the sphaleron transition rate $\Gamma_{\text{sphal.}}$, which controls the mean squared change in the axial charge in thermal equilibrium through the relation

$$\langle (\Delta Q_{A,q})^2 \rangle_{\text{therm.}} = 4Vt\Gamma_{\text{sphal.}} , \quad (107)$$

where V is the spatial volume and q denotes quark flavor. In ref. [78], it was noted that sphaleron transitions are not allowed in the boost invariant classical Glasma. Because the classical dynamics is effectively 2+1-dimensional, the second homotopy group of SU(3) gauge theory is zero, disallowing integer valued topological transitions. The quantum fluctuations we have been describing are no longer boost invariant, thereby allowing sphaleron transitions to go. In the non-equilibrium Glasma, the relevant quantity for computing the mean square change in the axial charge is the Wightman function [79]

$$G_{F\bar{F}}^>(X, Y) = \left\langle \frac{g^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}_b^{\mu\nu}(X) \frac{g^2}{32\pi^2} F_{\alpha\beta}^a \tilde{F}_b^{\alpha\beta}(Y) \right\rangle , \quad (108)$$

with $\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$. The sphaleron transition rate is defined in terms of this quantity as

$$\Gamma_{\text{sphal}} = \int d^4X G_{F\bar{F}}^>(X, 0) . \quad (109)$$

This quantity may be computed by a formula similar to eq. (106). It has aroused much interest recently because in semi-peripheral heavy ion collisions, the combined effect of large external electromagnetic \vec{B} fields and a large rate for topological transitions can lead to an induced charge separation phenomenon, called the *Chiral Magnetic effect* [80], with observable consequences. Our approach allows in principle an *ab initio* computation of this effect.

- *Jet quenching*

Besides the large flow observed in heavy ion collisions, the apparent strong modification of rare final states (such as jets, high p_\perp hadrons and heavy flavors) by the medium is adduced as confirmation of the high degree of opacity in the medium consistent with a strongly correlated fluid. The standard energy loss mechanism is radiative energy loss, which is dominated by collinear splittings that are primarily influenced by late time dynamics in a strongly interacting quark-gluon plasma. For a sampling of reviews, see refs. [81–84]. There are however potentially large modifications of the spectra of hard final states from energy loss at early times, that are not included in this energy loss scenario [85]; for a recent treatment of large angle contributions in a framework similar to ours, see ref. [86]. However, no computation has thus far included next-to-leading order corrections to, for example, the single inclusive parton spectrum at early times taking into account both small x evolution and multiple scattering effects. This is done in our formalism when $T^{\mu\nu}$ in eq. (106) is replaced by $E_p \frac{dN}{d^3\mathbf{p}}$; in fact, eq. (106) sums up a class of all order graphs that correspond to coherent multiple emissions where the momenta of all but one of the final state gluons

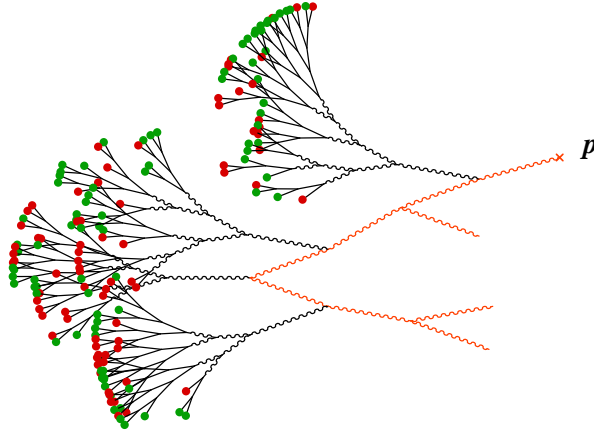


Figure 3: Example of graph included via the resummation of eq. (106). The green and red dots represent the color charges that describe the gluon content of the colliding nuclei in the CGC framework.

is integrated over. A typical contribution is illustrated in the figure 3. Note that this diagram, corresponding to the resummed case, looks like a parton shower interacting with the background Glasma fields. However, unlike vacuum showers which can in space-time be visualized as being logarithmically divergent in the proper time, these “showers” are a consequence of the exponentially growing contributions from leading instabilities at each order in perturbation theory.

We note that the single inclusive gluon spectrum also potentially suffers from collinear and infrared singularities. In the Glasma, it is possible that these could be regulated by strong multiple scattering and/or screening effects, but that remains to be proved. To avoid such complications, as in usual jet physics [87,88], one can look instead at correlators of the energy-momentum tensor that correspond to energy flow and are manifestly infrared and collinear safe. Clearly, there are a number of issues that need to be resolved here; the promising aspect of our formalism is that parton evolution, radiation and re-scattering can likely be treated, without ad hoc assumptions, in a consistent manner at early times.

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A Alternative basis for the free field solution

We will show in this appendix, that the form of the vacuum fluctuations can be written in terms of simple analytic functions if we trade the index ν for an index θ introduced via the following transformation

$$a_{\mathbf{k}_\perp}^\mu(\mathbf{x}) \equiv \int \frac{d\nu}{2\pi} e^{-i\nu\theta} a_{\mathbf{k}}^\mu(\mathbf{x}), \quad (110)$$

where we denote $\bar{\mathbf{k}} \equiv (\theta, \mathbf{k}_\perp)$.

To see that we are free to make this change in the quantum numbers, let us start from the real-valued field fluctuation as written in terms of the ν coordinate

$$a^\mu(x) = \sum_{\lambda=1,2} \int \frac{d^2\mathbf{k}_\perp d\nu}{(2\pi)^3} \left[c_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda}^\mu(x) + c_{\mathbf{k}\lambda}^* a_{\mathbf{k}\lambda}^{\mu*}(x) \right].$$

Then, making the trade from ν to θ we obtain

$$\begin{aligned} a^\mu(x) &= \sum_{\lambda=1,2} \int \frac{d^2\mathbf{k}_\perp d\nu}{(2\pi)^3} \left[c_{\mathbf{k}\lambda} \int d\theta e^{i\nu\theta} a_{\mathbf{k}\lambda}^\mu(x) + c_{\mathbf{k}\lambda}^* \int d\theta e^{i\nu\theta} a_{\mathbf{k}\lambda}^{\mu*}(x) \right] \\ &= \sum_{\lambda=1,2} \int \frac{d^2\mathbf{k}_\perp d\theta}{(2\pi)^3} \left[\int d\nu e^{i\nu\theta} c_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda}^\mu(x) + \int d\nu e^{i\nu\theta} c_{\mathbf{k}\lambda}^* a_{\mathbf{k}\lambda}^{\mu*}(x) \right] \\ &= \sum_{\lambda=1,2} \int \frac{d^2\mathbf{k}_\perp d\theta}{(2\pi)^3} \left[d_{\bar{\mathbf{k}}\lambda} a_{\bar{\mathbf{k}}\lambda}^\mu(x) + d_{\bar{\mathbf{k}}\lambda}^* a_{\bar{\mathbf{k}}\lambda}^{\mu*}(x) \right], \end{aligned}$$

where we defined

$$d_{\bar{\mathbf{k}}\lambda} \equiv \int d\nu c_{\mathbf{k}\lambda} e^{i\nu\theta}$$

and in keeping with the notation employed throughout the text we have used $\mathbf{k} \equiv (\nu, \mathbf{k}_\perp)$ and $\bar{\mathbf{k}} \equiv (\theta, \mathbf{k}_\perp)$. Since $c_{\mathbf{k}\lambda}$ is a Gaussian random variable so is its Fourier Transform, $d_{\bar{\mathbf{k}}\lambda}$, and we are free to generate random fluctuations using either basis.

Using this new basis, the first physical solutions transforms to

$$a_{\mathbf{k}_\perp}^\mu(\mathbf{x}) \equiv \int \frac{d\nu}{2\pi} e^{-i\nu\theta} a_{\mathbf{k}_\perp}^\mu(\mathbf{x}) = \frac{i}{2\sqrt{\pi}k_\perp} \begin{pmatrix} \mathbf{k}_y \\ -\mathbf{k}_x \\ 0 \end{pmatrix} e^{-ik_\perp \tau \cosh(\theta-\eta) + i\mathbf{k}_\perp \cdot \mathbf{x}_\perp}, \quad (111)$$

After the transformation to the θ variable, eqs. (82) can be integrated over time

$$\begin{aligned} \alpha_{2\perp} &= \frac{i}{2\sqrt{\pi}k_\perp} \tanh(\theta - \eta) e^{-ik_\perp \tau \cosh(\theta-\eta)} \\ \alpha_{2\eta} &= \frac{i}{2\sqrt{\pi}k_\perp} \left[\frac{1 + ik_\perp \tau \cosh(\theta - \eta)}{\cosh^2(\theta - \eta)} \right] e^{-ik_\perp \tau \cosh(\theta-\eta)}. \end{aligned} \quad (112)$$

Note that eqs. (81) only specify $\alpha_{2\perp}$ and $\alpha_{2\eta}$ up to a τ independent function. The τ independent functions that can be added to $\alpha_{2\perp}$ and $\alpha_{2\eta}$ are not independent, because this modification must correspond to a residual gauge transformation. Thus the allowed modifications are

$$\alpha_{2\perp} \rightarrow \alpha_{2\perp} + f, \quad \alpha_{2\eta} \rightarrow \alpha_{2\eta} + i(\partial_\theta f), \quad (113)$$

where f is an arbitrary τ independent function. We can choose to fix this residual gauge freedom by choosing the function f such that $a_{\mathbf{k}2}^\mu(\mathbf{x})$ vanishes at $\tau = 0^+$. After fixing the residual gauge freedom, we have the following expression for the second physical solution

$$a_{\mathbf{k}2}^\mu(\mathbf{x}) = \frac{i}{2\sqrt{\pi}k_\perp} \begin{pmatrix} \mathbf{k}_x g_\perp \\ \mathbf{k}_y g_\perp \\ g_\eta \end{pmatrix} e^{-ik_\perp \tau \cosh(\theta-\eta) + i\mathbf{k}_\perp \cdot \mathbf{x}_\perp}, \quad (114)$$

where we denote,

$$\begin{aligned} g_\perp &= i \tanh(\theta - \eta) \left[1 - e^{ik_\perp \tau \cosh(\theta-\eta)} \right] \\ g_\eta &= -\frac{1 - e^{ik_\perp \tau \cosh(\theta-\eta)} + ik_\perp \tau \cosh(\theta - \eta)}{\cosh^2(\theta - \eta)}. \end{aligned} \quad (115)$$

B Wightman functions for free fields

In this appendix, we shall construct the equal τ Wightman function for free fields. The equal-time Wightman function corresponding to the first physical solution $a_{\mathbf{k}1}^\mu$ is given by

$$G_1^{\mu\nu} \equiv \int \frac{d^2 \mathbf{k}_\perp}{(2\pi)^2} d\theta a_{\mathbf{k}1}^{\mu*}(\tau, \eta, \mathbf{x}_\perp) a_{\mathbf{k}1}^\nu(\tau, \eta', \mathbf{x}'_\perp). \quad (116)$$

Using the explicit expression in eq. (110), $G_1^{\mu\nu}$ can be written in the following formal way,

$$G_1^{ij} = \frac{1}{(2\pi)^2} \begin{pmatrix} -\partial_y^2 & \partial_y \partial_x \\ \partial_x \partial_y & -\partial_x^2 \end{pmatrix} \mathcal{F}_1 \left(i\partial_\eta, 2\sqrt{-\tau^2 \partial_\perp^2} \right) \delta(\eta - \eta') \ln \left(\frac{1}{\Lambda |x_\perp - x'_\perp|} \right), \quad (117)$$

where Λ is an infrared cutoff²⁴ and we have defined

$$\mathcal{F}_1(\nu, d_\perp) \equiv \frac{1}{2} \int \frac{dx}{\sqrt{1+x^2}} d\eta e^{i\nu\eta} e^{-ixd_\perp \sinh(\eta/2)}. \quad (118)$$

Likewise, the Wightman function corresponding to the second physical solution is given by

$$G_2^{\mu\nu} \equiv \int \frac{d^2 \mathbf{k}_\perp}{(2\pi)^2} d\theta a_{\mathbf{k}2}^{\mu*}(\tau, \eta, \mathbf{x}_\perp) a_{\mathbf{k}2}^\nu(\tau, \eta', \mathbf{x}'_\perp). \quad (119)$$

From the discussion following eq. (81), we noted that there was a residual gauge freedom remaining even after finding the two physical solutions. In the appendix A, we chose to fix this gauge by requiring that $\alpha_\eta \rightarrow 0$ at $\tau = 0^+$ in order that Hamilton's equations are regular at $\tau = 0$. Clearly, for the problem at hand, this is the correct choice. However, computing the Wightman function will be made much easier by choosing a different gauge,

$$g_\perp = i \tanh(\theta - \eta) \quad , \quad g_\eta = -\frac{1 + ik_\perp \tau \cosh(\theta - \eta)}{\cosh^2(\theta - \eta)}.$$

²⁴The final result does not depend on this cutoff, thanks to the derivatives acting on the logarithm.

We should stress that there is a residual gauge freedom remaining in any correlator of gauge fields. With this new gauge choice we find

$$G_2^{ij} = -\frac{1}{(2\pi)^2} \begin{pmatrix} \partial_x^2 & \partial_y \partial_x \\ \partial_x \partial_y & \partial_y^2 \end{pmatrix} \mathcal{F}_2 \left(i\partial_\eta, 2\sqrt{-\tau^2 \partial_\perp^2} \right) \delta(\eta - \eta') \ln \left(\frac{1}{\Lambda |x_\perp - x'_\perp|} \right), \quad (120)$$

with

$$\mathcal{F}_2(\nu, d_\perp) \equiv \frac{1}{2} \int \frac{dx}{\sqrt{1+x^2}} d\eta \left[1 - \frac{2 \cosh \eta}{1 + 2x^2 + \cosh \eta} \right] e^{i\nu\eta} e^{-ixd_\perp \sinh(\eta/2)}. \quad (121)$$

We can also calculate the correlation function of the canonical momenta defined by $e^i = \tau \partial_\tau a_i$. The free Wightman function for the transverse components of the electric field is defined as

$$H^{ij} = \sum_{\lambda=1,2} \tau^2 \int \frac{d^2 \mathbf{k}_\perp}{(2\pi)^2} d\theta \left[\partial_\tau a_{\mathbf{k}\lambda}^{i*}(\tau, \eta, \mathbf{x}_\perp) \right] \left[\partial_\tau a_{\mathbf{k}\lambda}^j(\tau, \eta', \mathbf{x}'_\perp) \right]. \quad (122)$$

Note that the τ derivatives remove the residual gauge freedom that still remained in the expression for G^{ij} as written above. This is expected since we don't expect any residual gauge freedom to remain when computing correlators of physical quantities such as the electric field. The Wightman function for the first physical solution is

$$H_1^{ij} = \frac{1}{(2\pi)^2} \begin{pmatrix} -\partial_y^2 & \partial_y \partial_x \\ \partial_x \partial_y & -\partial_x^2 \end{pmatrix} \mathcal{F}_3 \left(i\partial_\eta, 2\sqrt{-\tau^2 \partial_\perp^2} \right) \delta(\eta - \eta') \ln \left(\frac{1}{\Lambda |x_\perp - x'_\perp|} \right), \quad (123)$$

and similarly for the second physical solution

$$H_2^{ij} = -\frac{1}{(2\pi)^2} \begin{pmatrix} \partial_x^2 & \partial_y \partial_x \\ \partial_x \partial_y & \partial_y^2 \end{pmatrix} \mathcal{F}_4 \left(i\partial_\eta, 2\sqrt{-\tau^2 \partial_\perp^2} \right) \delta(\eta - \eta') \ln \left(\frac{1}{\Lambda |x_\perp - x'_\perp|} \right), \quad (124)$$

where the functions $\mathcal{F}_{3,4}$ are defined as

$$\mathcal{F}_3(\nu, d_\perp) \equiv \frac{1}{4} \int \frac{dx}{\sqrt{1+x^2}} d\eta \left[1 + 2x^2 + \cosh \eta \right] e^{i\nu\eta} e^{-ixd_\perp \sinh(\eta/2)}, \quad (125)$$

$$\mathcal{F}_4(\nu, d_\perp) \equiv \frac{1}{4} \int \frac{dx}{\sqrt{1+x^2}} d\eta \left[1 + 2x^2 - \cosh \eta \right] e^{i\nu\eta} e^{-ixd_\perp \sinh(\eta/2)}. \quad (126)$$

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